

Dissipation in relativistic superfluid neutron stars

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ABSTRACT

We analyze damping of oscillations of general relativistic superfluid neutron stars. To this aim we extend the method of decoupling of superfluid and normal oscillation modes first suggested in [Gusakov & Kantor PRD 83, 081304(R) (2011)]. All calculations are made self-consistently within the finite temperature superfluid hydrodynamics. The general analytic formulas are derived for damping times due to the shear and bulk viscosities. These formulas describe both normal and superfluid neutron stars and are valid for oscillation modes of arbitrary multipolarity. We show that: (i) use of the ordinary one-fluid hydrodynamics is a good approximation, for most of the stellar temperatures, if one is interested in calculation of the damping times of normal f -modes; (ii) for radial and p -modes such an approximation is poor; (iii) the temperature dependence of damping times undergoes a set of rapid changes associated with resonance coupling of neighboring oscillation modes. The latter effect can substantially accelerate viscous damping of normal modes in certain stages of neutron-star thermal evolution.

Key words: stars: neutron – stars: oscillations – stars: interiors.

1 INTRODUCTION

Neutron stars (NS) are compact objects with the mass $M \sim M_\odot$, circumferential radius $R \sim 10$ km, and the central density ρ_c several times higher than the nuclear density $\rho_0 \approx 2.8 \times 10^{14}$ g cm⁻³. They are interesting because of extreme conditions in their interiors and a wide variety of associated astrophysical phenomena. In particular, internal instabilities or external perturbations can excite NS oscillations, which are potentially detectable by the next-generation gravitational wave interferometers (see, e.g., Andersson & Kokkotas 2001; Andersson 2003; Owen 2010). It is very probable, that quasiperiodic oscillations of electromagnetic radiation observed in the tails of the giant gamma-ray flares are connected with oscillations in NS crust (e.g., Israel et al. 2005; Strohmayer & Watts 2005, 2006; Watts & Strohmayer 2007), and that seismology would become a significant source of information about NSs in the nearest future (Abbot et al. 2007; Watts 2011; Andersson et al. 2011).

For the correct interpretation of already existing and future observations one requires a well-developed theory of oscillating NSs. It should, in particular: (i) be based on the general relativity theory, since NSs are relativistic objects; (ii) employ an adequate model of superdense matter, including realistic equation of state and parameters of baryon superfluidity; (iii) correctly account for the effects of baryon superfluidity on the hydrodynamics of NS matter.

Let us discuss briefly a (key) role of superfluidity. According to numerous microscopic calculations (see, e.g., Lombardo & Schulze 2001), baryon matter in the internal layers of neutron stars becomes superfluid at $T \lesssim 10^8$ – 10^{10} K. It is very difficult to interpret the observational data on pulsar glitches (see, e.g., Chamel & Haensel 2008) and cooling of NSs (Yakovlev, Levenfish & Shibano

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1999; Yakovlev & Pethick 2004) without invoking baryon superfluidity. Recent real-time observations of cooling NS in Cassiopea A supernova remnant (Heinke & Ho 2010) also present a strong argument in favor of the existence of baryon superfluidity in the NS core. The observations were explained by Shternin et al. (2011) and Page et al. (2011) within a scenario, suggested for the first time in Gusakov et al. (2004) and Page et al. (2004), and assuming mild neutron superfluidity (with maximum neutron critical temperatures $T_{\text{cn max}} \sim 7 \div 9 \times 10^8$ K) and strong proton superconductivity (with maximum proton critical temperatures $T_{\text{cp max}} \gtrsim 2 \div 3 \times 10^9$ K) in the NS core.

Combined analysis of all the three factors (i)–(iii) is a formidable task for the oscillation theory so in the literature they were considered successively. The foundations of the relativistic theory of stellar oscillations were laid fifty years ago by Chandrasekhar (1964) and Thorne & Campolattaro (1967) and were further developed in many subsequent papers (see, e.g., Ipser & Thorne 1973; Detweiler & Ipser 1973; Lindblom & Detweiler 1983; Detweiler & Lindblom 1985; Cutler & Lindblom 1987; Cutler, Lindblom & Splinter 1990; Chandrasekhar & Ferrari 1991; Kokkotas & Schutz 1992; Yoshida & Lee 2003a; Lin, Andersson & Comer 2008 and a review of Kokkotas & Schmidt 1999). When studying oscillations of NSs, most of these works considered ordinary one-fluid relativistic hydrodynamics.

Meanwhile it is well known that superfluidity leads to appearance of additional velocity fields, describing the superfluid degrees of freedom (e.g., Khalatnikov & Lebedev 1982; Khalatnikov 1989; Carter & Khalatnikov 1992). This substantially complicates the hydrodynamics of NS matter, making it multi-fluid (Mendell 1991a,b; Gusakov & Andersson 2006). In addition, superfluidity affects the kinetic coefficients (such as bulk and shear viscosities) and also requires additional viscous coefficients to be introduced (see Gusakov 2007; Gusakov & Kantor 2008 for details).

Oscillations of superfluid NSs have been studied actively only in the last two decades (see, e.g., Lee 1995; Lindblom & Mendell 2000; Prix & Rieutord 2002; Yoshida & Lee 2003b; Prix, Comer & Andersson 2004; Samuelsson & Andersson 2009; Wong, Lin & Leung 2009; Passamonti & Andersson 2011, 2012), starting from the pioneering papers by Epstein (1988) and Lindblom & Mendell (1994). However, most of these works neglect general relativity effects and employ zero temperature ($T = 0$) limit of superfluid hydrodynamics (i.e., hydrodynamics, applicable only at $T = 0$). Within the general relativity theory oscillations were discussed only by Comer, Langlois & Lin (1999); Andersson, Comer & Langlois (2002); Yoshida & Lee (2003a); Gusakov & Andersson (2006); Lin et al. (2008); Kantor & Gusakov (2011); Chugunov & Gusakov (2011), but most of these works used zero temperature hydrodynamics. Moreover, in some of these papers (e.g., Andersson et al. 2002; Lin et al. 2008) the presence of superfluid component was modeled by an artificial (polytropic) equation of state which does not represent any specific microphysical model.

Only in the recent papers Gusakov & Andersson (2006); Kantor & Gusakov (2011); Chugunov & Gusakov (2011) an attempt was made to self-consistently calculate the oscillation spectra using a realistic model of superdense matter and allowing for the effects of finite stellar temperatures. It was shown that in many cases an approximation $T = 0$ is not justified and, moreover, it can lead to *qualitatively incorrect* results (Kantor & Gusakov 2011; Chugunov & Gusakov 2011).

Of particular interest is the question of how superfluidity influences dissipation of neutron star oscillations. It is of extreme importance, for instance, for understanding physical conditions under which a rotating NS becomes unstable with respect to excitation of various oscillations (e.g., r -modes), and for estimating gravitational radiation from such stars (e.g., Andersson & Kokkotas 2001).

There were several serious and successful attempts to allow for the effects of superfluidity when studying the dissipation of oscillations in NSs (see, e.g., Lindblom & Mendell 1995, 2000; Lee & Yoshida 2003; Haskell, Andersson & Passamonti 2009; Andersson, Glampedakis & Haskell 2009; Haskell & Andersson 2010; Passamonti & Glampedakis 2012), but *all of them* considered Newtonian stars and used the $T = 0$ superfluid hydrodynamics. The self-consistent analysis of dissipation in superfluid NSs was only recently performed for a simple case of a radially oscillating NS (Kantor & Gusakov 2011).

The aim of the present paper is to fill this gap and to consider, for the first time, dissipation of nonradial oscillations in general relativistic superfluid NSs employing realistic microphysics input with accurate treatment of the effects of finite stellar temperatures.

The paper is organized as follows. Relativistic superfluid hydrodynamics is briefly reviewed in Sec. 2. Sec. 3 discusses an unperturbed star and introduces variables describing small deviations of NS from equilibrium. In Sec. 4 we derive expressions for the oscillation energy and its dissipation rates due to bulk and shear viscosities. In Sec. 5 the equations that govern oscillations of superfluid NSs are explicitly written out. Sec. 6 describes the approach to study dissipation of superfluid NS oscillations. This approach is applied for a detailed numerical analysis of realistic models of oscillating neutron stars in Sec. 7. Sec. 8 presents a summary of our results.

In what follows, we use the system of units in which $c = k_B = 1$, where c is the speed of light and k_B is the Boltzmann constant.

2 DISSIPATIVE SUPERFLUID HYDRODYNAMICS

In this paper we consider, for simplicity, npe-matter in NS cores, that is matter composed of neutrons (n), protons (p), and electrons (e). Because both protons and neutrons can be in the superfluid state, one has to use the relativistic hydrodynamics of superfluid mixtures to study oscillations of NSs. Here we briefly discuss the corresponding equations to establish notations and to make the presentation more self-contained. Our consideration closely follows the papers by Gusakov & Andersson (2006); Gusakov (2007) and, especially, Kantor & Gusakov (2011). The reader is referred to these works for more details.

The main distinctive feature of superfluid hydrodynamics is the presence of several velocity fields in the mixture. In our case, these are the four-velocity u^μ of the ‘normal’ (nonsuperfluid) component of matter (electrons and Bogoliubov excitations of neutrons and protons) as well as the ‘four-velocities’ of superfluid neutrons $v_{s(n)}^\mu$ and superfluid protons $v_{s(p)}^\mu$. In what follows instead of the velocities $v_{s(n)}^\mu$ and $v_{s(p)}^\mu$ it will be convenient to use the four-vectors $w_{(i)}^\mu = \mu_i[v_{s(i)}^\mu - u^\mu]$, where μ_i is the relativistic chemical potential for particle species $i = n$ or p . A presence of several velocity fields modifies the expressions for the current densities of neutrons $j_{(n)}^\mu$ and protons $j_{(p)}^\mu$,

$$j_{(i)}^\mu = n_i u^\mu + Y_{ik} w_{(k)}^\mu \quad (1)$$

in comparison with the standard expression $j_{(i)}^\mu = n_i u^\mu$. The electron current density $j_{(e)}^\mu$ has a standard form,

$$j_{(e)}^\mu = n_e u^\mu. \quad (2)$$

Here and below the subscripts i and k refer to nucleons: $i, k = n, p$; n_l is the number density of particle species $l = n, p, e$. Unless otherwise stated the summation is assumed over the repeated nucleon indices i, k and over the spacetime indices μ, ν, \dots (Greek letters). In Eq. (1) Y_{ik} is the relativistic entrainment matrix, which is a generalization of the concept of superfluid density (see, e.g., Khalatnikov 1989) to the case of relativistic mixtures. In the nonrelativistic theory, a similar matrix was first considered by Andreev & Bashkin (1975). The matrix Y_{ik} is symmetric, $Y_{ik} = Y_{ki}$, and is expressed in terms of the Landau parameters F_1^{ik} of asymmetric nuclear matter and universal functions of temperature, Φ_i , as described in Gusakov, Kantor & Haensel (2009b). In beta-equilibrium it can be presented as a function of density ρ and the combinations T/T_{cn} and T/T_{cp} : $Y_{ik} = Y_{ik}(\rho, T/T_{cn}, T/T_{cp})$, where T is the temperature; $T_{cn}(\rho)$ and $T_{cp}(\rho)$ are the density-dependent neutron and proton critical temperatures, respectively. If, for example, $T > T_{cn}$ then all neutrons are normal. The important property of the matrix Y_{ik} is that for any nonsuperfluid species $l = n$ or p , the corresponding elements Y_{lk} of this matrix vanish.

In the present paper we consider NS oscillations, whose frequencies are well below the electron and proton plasma frequencies. In that case the quasineutrality condition, $n_e = n_p$, should hold in an oscillating star, from which it follows (for a nonrotating non-magnetized NS) $j_{(p)}^\mu = j_{(e)}^\mu$ or, in view of (1) and (2),

$$Y_{pk} w_{(k)}^\mu = 0. \quad (3)$$

Below we assume that this condition is always satisfied. It relates the four-vectors $w_{(n)}^\mu$ and $w_{(p)}^\mu$.

In what follows, along with u^μ and $w_{(i)}^\mu$ it will be convenient to introduce the quantity X^μ , describing superfluid degrees of freedom, as well as the quantity which we call the ‘baryon four-velocity’ $U_{(b)}^\mu$ (notice, however, that it is not a four-velocity in the usual sense, because generally $U_{(b)}^\mu U_{(b)\mu} \neq -1$, see Eq. (45) and the footnote ⁴ below). They are defined by the formulas

$$X^\mu = \frac{Y_{nk} w_{(k)}^\mu}{n_b}, \quad (4)$$

$$U_{(b)}^\mu = u^\mu + X^\mu, \quad (5)$$

where $n_b = n_n + n_p$ is the baryon number density. Notice that, as follows from Eqs. (1)–(3), the baryon current density $j_{(b)}^\mu = j_{(n)}^\mu + j_{(p)}^\mu$ is related to $U_{(b)}^\mu$ by the standard equation,

$$j_{(b)}^\mu = n_b U_{(b)}^\mu, \quad (6)$$

while $j_{(e)}^\mu$ equals

$$j_{(e)}^\mu = n_e [U_{(b)}^\mu - X^\mu]. \quad (7)$$

Together with the quasineutrality condition ($n_e = n_p$) and Eq. (3), the equations of superfluid hydrodynamics include (Gusakov 2007):

(i) Continuity equations for baryons (b) and electrons (e),

$$j_{(b); \mu}^\mu = 0, \quad (8)$$

$$j_{(e); \mu}^\mu = 0; \quad (9)$$

(ii) Energy-momentum conservation

$$T_{;\mu}^{\mu\nu} = 0, \quad (10)$$

$$T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P g^{\mu\nu} + Y_{ik} \left(w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu \right) + \tau^{\mu\nu}, \quad (11)$$

$$\tau^{\mu\nu} = -\eta H^{\mu\gamma} H^{\nu\delta} \left(u_{\gamma;\delta} + u_{\delta;\gamma} - \frac{2}{3} g_{\gamma\delta} u_{;\varepsilon}^\varepsilon \right) - \xi_{1n} H^{\mu\nu} \left[Y_{nk} w_{(k)}^\gamma \right]_{;\gamma} - \xi_2 H^{\mu\nu} u_{;\gamma}^\gamma; \quad (12)$$

(iii) Potentiality condition for superfluid motion of neutrons

$$\partial_\nu \left[w_{(n)\mu} + (\mu_n + \varkappa_n) u_\mu \right] = \partial_\mu \left[w_{(n)\nu} + (\mu_n + \varkappa_n) u_\nu \right], \quad (13)$$

$$\varkappa_n = -\xi_{3n} \left[Y_{nk} w_{(k)}^\mu \right]_{;\mu} - \xi_{4n} u_{;\mu}^\mu; \quad (14)$$

as well as (iv) the second law of thermodynamics

$$d\varepsilon = T dS + \mu_e dn_e + \mu_i dn_i + \frac{Y_{ik}}{2} d(w_{(i)}^\alpha w_{(k)\alpha}). \quad (15)$$

In formulas (8)–(15) $g^{\mu\nu}$ is the metric tensor; $H^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$; $\partial_\mu \equiv \partial/(\partial x^\mu)$; P , ε , S , and μ_e are the pressure, energy density, entropy density, and relativistic electron chemical potential, respectively. These quantities are related by the formula

$$P = -\varepsilon + \mu_e n_e + \mu_i n_i + T S. \quad (16)$$

Finally, η is the shear viscosity coefficient and ξ_{1n} , ξ_2 , ξ_{3n} , ξ_{4n} are the bulk viscosity coefficients. Because of the Onsager symmetry principle, one has

$$\xi_{1n} = \xi_{4n}. \quad (17)$$

Moreover, if the bulk viscosities are generated solely by the direct or modified URCA processes, one has an additional constraint (Gusakov 2007)

$$\xi_{1n}^2 = \xi_2 \xi_{3n}. \quad (18)$$

In the absence of superfluidity the only nonzero coefficient is ξ_2 – the ordinary bulk viscosity.

To close the system describing superfluid hydrodynamics one should put two additional constraints on the four-vectors u^μ and $w_{(n)}^\mu$,

$$u_\mu u^\mu = -1, \quad (19)$$

$$u_\mu w_{(n)}^\mu = 0. \quad (20)$$

The first constraint is the standard normalization condition while the second one indicates that the comoving frame, in which we measure various thermodynamic quantities (e.g., n_i , ε , ...), is defined by the condition $u^\mu = (1, 0, 0, 0)$ (Gusakov & Andersson 2006; Gusakov 2007). Using Eqs. (1), (11), (12), (19), and (20) one then immediately finds that $n_l = -u_\mu j_{(l)}^\mu$ ($l = n, p, e$) and $\varepsilon = u_\mu u_\nu T^{\mu\nu}$.

Making use of the hydrodynamics described above, one can derive the entropy generation equation, valid for superfluid matter. Following the derivation of the similar equation (33) in Gusakov (2007), one arrives at

$$S_{;\mu}^\mu = -\frac{\varkappa_n}{T} \left[Y_{nk} w_{(k)}^\mu \right]_{;\mu} - \tau^{\mu\nu} \left(\frac{u_\nu}{T} \right)_{;\mu} \quad (21)$$

where the entropy density current S^μ is ¹

$$S^\mu = S u^\mu - \frac{u_\nu}{T} \tau^{\mu\nu}. \quad (22)$$

¹ Notice that, in Gusakov (2007) there is an additional term in the expression for S^μ , so that

$$S^\mu = S u^\mu - \frac{u_\nu}{T} \tau^{\mu\nu} - \frac{\varkappa_n}{T} Y_{nk} w_{(k)}^\mu.$$

The last term here appears naturally in the entropy generation equation. However, strictly speaking, it is small and should be neglected if one takes into account only the *largest* dissipative terms in the equations of superfluid hydrodynamics (this is the standard approximation; see Gusakov 2007 and §140 of Landau & Lifshitz 1987 for an explanation of what we mean by the ‘largest terms’). It remains to note that the terms similar to the last term in the expression for S^μ also appear in the most general form of the nonrelativistic superfluid dissipative hydrodynamics formulated by Clark (for details see the book by Putterman 1974).

When writing (21) we neglected small dissipative terms, as it is discussed in Gusakov (2007). Introducing

$$Q_{\text{bulk}} \equiv \left\{ \sqrt{\xi_{3n}} \left[Y_{nk} w_{(k)}^\mu \right]_{;\mu} + \sqrt{\xi_2} u_{;\mu}^\mu \right\}^2, \quad (23)$$

$$\begin{aligned} Q_{\text{shear}} &\equiv \eta H^{\mu\gamma} H^{\nu\delta} \left(u_{\gamma;\delta} + u_{\delta;\gamma} - \frac{2}{3} g_{\gamma\delta} u_{;\varepsilon}^\varepsilon \right) u_{\nu;\mu} \\ &= \frac{\eta}{2} H^{\mu\gamma} H^{\nu\delta} \left(u_{\gamma;\delta} + u_{\delta;\gamma} - \frac{2}{3} g_{\gamma\delta} u_{;\varepsilon}^\varepsilon \right) \left(u_{\nu;\mu} + u_{\mu;\nu} - \frac{2}{3} g_{\mu\nu} u_{;\varepsilon}^\varepsilon \right), \end{aligned} \quad (24)$$

Eq. (21) can be rewritten as

$$T(Su^\mu)_{;\mu} = Q_{\text{bulk}} + Q_{\text{shear}}. \quad (25)$$

To derive Eq. (25) we used Eqs. (17) and (18), as well as the fact that for the tensor (12) $\tau^{\mu\nu} u_\nu = 0$ ².

3 BASIC EQUATIONS

3.1 An unperturbed star

An equilibrium configuration of a nonrotating superfluid NS was analyzed in detail in section 3 of Gusakov & Andersson (2006). Here we present only the main results of this analysis, which will be used in what follows.

The metric of a spherically symmetric, nonrotating NS in equilibrium has the form

$$-ds^2 \equiv g_{\alpha\beta}^{(0)} dx^\alpha dx^\beta = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (26)$$

where r , θ , and φ , are the spatial coordinates in the spherical frame with the origin at the stellar centre; t is the time coordinate; $\nu(r)$ and $\lambda(r)$ are the metric coefficients for an unperturbed star.

The four-velocity u^μ , generally defined as

$$u^\mu = \frac{dx^\mu}{ds}, \quad (27)$$

in equilibrium equals

$$u^0 = e^{-\nu/2}, \quad u^1 = u^2 = u^3 = 0. \quad (28)$$

We assume that in the unperturbed star superfluid components are at rest with respect to the normal component. In that case the four-vectors $w_{(i)}^\mu$ satisfy

$$w_{(n)}^\mu = w_{(p)}^\mu = 0. \quad (29)$$

Using Eqs. (4), (5), (28), and (29), one has for the baryon four-velocity

$$U_{(b)}^0 = e^{-\nu/2}, \quad U_{(b)}^1 = U_{(b)}^2 = U_{(b)}^3 = 0. \quad (30)$$

In addition, the following conditions of hydrostatic equilibrium must hold for an unperturbed star,

$$\frac{dP}{dr} = -\frac{1}{2} (P + \varepsilon) \frac{d\nu}{dr}, \quad (31)$$

$$\frac{d}{dr} \left(\mu_n e^{\nu/2} \right) = 0. \quad (32)$$

The last condition should be only used in the stellar region where neutrons are superfluid (hereafter the SFL-region). One can show (Gusakov & Andersson 2006), that if an unperturbed NS is additionally in beta-equilibrium, that is, the imbalance $\delta\mu$ of chemical potentials vanishes,

$$\delta\mu \equiv \mu_n - \mu_p - \mu_e = 0, \quad (33)$$

then the SFL-region must also be in thermal equilibrium, with the redshifted internal stellar temperature T^∞ constant over this region,

$$T^\infty \equiv T e^{\nu/2} = \text{constant}. \quad (34)$$

In what follows we assume that the conditions (33) and (34) are satisfied in the entire core of the unperturbed NS. In the latter case Eq. (16) for the equilibrium pressure can be rewritten as

$$P = -\varepsilon + \mu_n n_b + TS. \quad (35)$$

It should also be stressed that, as long as we neglected the temperature effects when calculating the equilibrium stellar model,

² This equality holds true only if one neglects the thermal conductivity, as we assume in Eq. (12).

the hydrostatic structure of the unperturbed superfluid NS is indistinguishable from that of the normal (nonsuperfluid) star of the same mass.

3.2 Small departures from equilibrium

The metric of a perturbed star can be presented in the form

$$-ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta = (g_{\alpha\beta}^{(0)} + \delta g_{\alpha\beta}) dx^\alpha dx^\beta. \quad (36)$$

From here on the symbol δ denotes Eulerian perturbations, so that $\delta g_{\alpha\beta}$ corresponds to small metric perturbations in the course of stellar oscillations.

Since we study oscillations of a nonrotating nonmagnetized NS and neglect the effects of crystalline crust, all the perturbations in the system are of even parity³. In that case, in the appropriately chosen gauge $\delta g_{\alpha\beta} dx^\alpha dx^\beta$ can be written as (we follow the notations of Cutler et al. 1990)

$$\begin{aligned} \delta g_{\alpha\beta} dx^\alpha dx^\beta = & -e^\nu r^l H_0(r) Y_l^m e^{i\omega t} dt^2 - 2i\omega r^{l+1} H_1(r) Y_l^m e^{i\omega t} dt dr \\ & -e^\lambda r^l H_2(r) Y_l^m e^{i\omega t} dr^2 - r^{l+2} K(r) Y_l^m e^{i\omega t} (d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \quad (37)$$

In Eq. (37) we assumed that all the perturbations depend on t as $e^{i\omega t}$. In addition, we already expanded the perturbations into series in spherical harmonics Y_l^m , and consider a single harmonic with fixed l and m . The unknown functions H_0 , H_1 , H_2 , and K depend on r only, and should be determined from the linearized Einstein equations, describing NS oscillations (see Sec. 5). Depending on l the gauge of the metric can be further specialized (e.g., Cutler et al. 1990). Namely, one can choose the gauge such that for $l = 0$ (radial oscillations) $H_1 = K = 0$; for $l = 1$ (dipole oscillations) $K = 0$; for $l \geq 2$ $H_0 = H_2$.

As follows from the definition (27), in the perturbed star the four-velocity u^μ of the normal component equals, in the linear approximation

$$u^0 = \frac{1}{\sqrt{-g_{00}}} = e^{-\nu/2} \left(1 - \frac{1}{2} r^l H_0 Y_l^m e^{i\omega t} \right), \quad (38)$$

$$u^j = v^j e^{-\nu/2}, \quad (39)$$

where

$$v^j \equiv \frac{dx^j}{dt} \quad (40)$$

is the j -th component of the velocity of the normal liquid. Here and below j is the spatial index, $j = r, \theta$, and φ . Similarly, using Eqs. (19) and (20) one can show that for small deviations from equilibrium

$$w_{(i)}^0 = 0, \quad (41)$$

while the spatial components $w_{(i)}^j$ are small quantities, linear in perturbation (for a similar consideration see Gusakov & Andersson 2006). In what follows, instead of the four-vectors $w_{(i)}^\mu$ [which are constrained by Eq. (3)] it will be often more convenient to use the quantity X^μ , defined by (4). For small perturbations

$$X^0 = 0, \quad (42)$$

while X^j is non-zero but small (linear in perturbations).

Using Eqs. (38), (39) and (42), as well as the definition (5), it is easy to write out an expression for the baryon four-velocity $U_{(b)}^\mu$ in the perturbed star,

$$U_{(b)}^0 = \frac{1}{\sqrt{-g_{00}}} = e^{-\nu/2} \left(1 - \frac{1}{2} r^l H_0 Y_l^m e^{i\omega t} \right), \quad (43)$$

$$U_{(b)}^j = v_{(b)}^j e^{-\nu/2}, \quad (44)$$

where the last equality is the definition of the j -th component of the baryon velocity $v_{(b)}^j$ (linear in perturbation). Notice that, as follows from Eqs. (43) and (44), in the linear approximation the normalization condition for the baryon four-velocity is the same

$$U_{(b)\mu} U_{(b)}^\mu = -1, \quad (45)$$

as for u^μ ⁴. In what follows instead of the velocities v^j and $v_{(b)}^j$, it will be more convenient to use the corresponding *Lagrangian*

³ A more detailed argument can be found in Thorne & Campolattaro (1967); see also Regge & Wheeler (1957).

⁴ However, beyond the linear approximation, Eqs. (4), (5), (19), and (20) yield $U_{(b)\mu} U_{(b)}^\mu = -1 + Y_{ni} Y_{nk} w_{(i)\mu} w_{(k)}^\mu / n_b^2$. The normalization condition (45) is generally not fulfilled because the reference frame in which $U_{(b)}^\mu = (1, 0, 0, 0)$ is not comoving [that is, $j_{(b)}^\mu U_{(b)\mu} \neq -n_b$ in that reference frame]. As it was already indicated in Sec. 2, all thermodynamic variables are measured in the reference frame, in which $u^\mu = (1, 0, 0, 0)$.

displacements. They are defined by the equalities

$$v^j \equiv \frac{\partial \xi^j}{\partial t} = i\omega \xi^j, \quad (46)$$

$$v_{(b)}^j \equiv \frac{\partial \xi_{(b)}^j}{\partial t} = i\omega \xi_{(b)}^j. \quad (47)$$

Introducing also the analogue of the Lagrangian displacement $\xi_{(\text{sfl})}^j$ for the vector X^j , one can write

$$X^j \equiv e^{-\nu/2} \frac{\partial \xi_{(\text{sfl})}^j}{\partial t} = i\omega e^{-\nu/2} \xi_{(\text{sfl})}^j. \quad (48)$$

In terms of the Lagrangian displacements the equality (5) can be presented as

$$\xi_{(b)}^j = \xi^j + \xi_{(\text{sfl})}^j. \quad (49)$$

Because of the spherical symmetry of the unperturbed star it is sufficient to consider Lagrangian displacements ξ^j , $\xi_{(b)}^j$, and $\xi_{(\text{sfl})}^j$, of the form [see also a note after Eq. (90) below]

$$\xi^j = [\xi^r, \xi^\theta, \xi^\varphi] = [e^{-\lambda/2} r^{l-1} W(r) Y_l^0, -r^{l-2} V(r) \partial_\theta Y_l^0, 0] e^{i\omega t}, \quad (50)$$

$$\xi_{(b)}^j = [\xi_{(b)}^r, \xi_{(b)}^\theta, \xi_{(b)}^\varphi] = [e^{-\lambda/2} r^{l-1} W_b(r) Y_l^0, -r^{l-2} V_b(r) \partial_\theta Y_l^0, 0] e^{i\omega t}, \quad (51)$$

$$\xi_{(\text{sfl})}^j = [\xi_{(\text{sfl})}^r, \xi_{(\text{sfl})}^\theta, \xi_{(\text{sfl})}^\varphi] = [e^{-\lambda/2} r^{l-1} W_{\text{sfl}}(r) Y_l^0, -r^{l-2} V_{\text{sfl}}(r) \partial_\theta Y_l^0, 0] e^{i\omega t}, \quad (52)$$

where W , V , W_b , V_b , W_{sfl} , and V_{sfl} are some functions of r to be derived from oscillation equations. In Eqs. (50)–(52) $Y_l^0 = \sqrt{(2l+1)/(4\pi)} P_l(\cos \theta)$, where P_l is the Legendre polynomial. Here and below we consider only spherical harmonics with $m = 0$. We can do this without any loss of generality, because, due to the spherical symmetry of the unperturbed star, oscillation eigenfrequencies as well as eigenfunctions H_0 , H_1, \dots , W_{sfl} , and V_{sfl} , introduced in this section, cannot depend on m (see, e.g., Thorne & Campolattaro 1967).

It follows from Eqs. (49) and (50)–(52) that

$$W_b = W + W_{\text{sfl}}, \quad (53)$$

$$V_b = V + V_{\text{sfl}}. \quad (54)$$

4 DAMPING OF OSCILLATIONS DUE TO THE BULK AND SHEAR VISCOSITIES: GENERAL FORMULAS

In the present paper among the possible mechanisms of dissipation of oscillation energy we take into account damping due to the bulk and shear viscosities as well as due to radiation of gravitational waves. Dissipation makes the oscillation frequency ω complex, so that it can be presented in the form,

$$\omega = \sigma + \frac{i}{\tau}, \quad (55)$$

where σ is the real part of the frequency, and τ is the characteristic damping time. Assuming that damping is weak, in the linear approximation one can present the following standard expression for τ ,

$$\frac{1}{\tau} = -\frac{1}{2E_{\text{mech}}} \frac{dE_{\text{mech}}}{dt}, \quad (56)$$

where E_{mech} is the mechanical energy of oscillations; dE_{mech}/dt is the dissipation rate of the mechanical energy, which can be presented as

$$\frac{dE_{\text{mech}}}{dt} = -\mathfrak{W}_{\text{bulk}} - \mathfrak{W}_{\text{shear}} - \mathfrak{W}_{\text{grav}}, \quad (57)$$

where $\mathfrak{W}_{\text{bulk}}$, $\mathfrak{W}_{\text{shear}}$, and $\mathfrak{W}_{\text{grav}}$ are the energy, dissipated per unit time due to the bulk viscosity, shear viscosity, and gravitational radiation, respectively. Introducing partial damping times τ_{bulk} , τ_{shear} , and τ_{grav} according to

$$\frac{1}{\tau_{\text{bulk}}} = \frac{\mathfrak{W}_{\text{bulk}}}{2E_{\text{mech}}}, \quad (58)$$

$$\frac{1}{\tau_{\text{shear}}} = \frac{\mathfrak{W}_{\text{shear}}}{2E_{\text{mech}}}, \quad (59)$$

$$\frac{1}{\tau_{\text{grav}}} = \frac{\mathfrak{W}_{\text{grav}}}{2E_{\text{mech}}}, \quad (60)$$

one can rewrite the expression for τ as

$$\frac{1}{\tau} = \frac{1}{\tau_{\text{grav}}} + \frac{1}{\tau_{\text{shear}}} + \frac{1}{\tau_{\text{bulk}}}. \quad (61)$$

Thus, to calculate τ we need to know the mechanical energy E_{mech} of NS oscillations, as well as the quantities $\mathfrak{W}_{\text{bulk}}$, $\mathfrak{W}_{\text{shear}}$, and $\mathfrak{W}_{\text{grav}}$.

4.1 Mechanical energy

The general relativistic expression for the mechanical energy of oscillating normal (nonsuperfluid) NS was obtained by Thorne & Campolattaro (1967) (see also Meltzer & Thorne 1966). Their result can be easily generalized to the case of superfluid matter. Mechanical energy E_{mech} is related to the averaged over the oscillation period $2\pi/\sigma$ kinetic energy \bar{E}_{kin} by the standard formula,

$$E_{\text{mech}} = 2\bar{E}_{\text{kin}}. \quad (62)$$

Thus, to determine E_{mech} one needs to know E_{kin} . One can write (e.g., Thorne & Campolattaro 1967)

$$E_{\text{kin}} = \int_{\text{star}} \epsilon_{\text{kin}} e^{\nu/2} dV, \quad (63)$$

where $dV = r^2 e^{\lambda/2} \sin\theta d\theta d\varphi dr$ is the proper volume element; ϵ_{kin} is the kinetic energy density measured in the locally flat coordinate system \tilde{x}^μ [with the metric $-ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$, where $\tilde{g}_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$], which is at rest with respect to the unperturbed star. If the NS matter is normal, ϵ_{kin} is given by

$$\epsilon_{\text{kin}} = \frac{1}{2} (P + \varepsilon) \left[(\tilde{u}^r)^2 + (\tilde{u}^\theta)^2 + (\tilde{u}^\varphi)^2 \right]. \quad (64)$$

Here

$$\tilde{u}^j = \frac{\partial \tilde{x}^j}{\partial x^\mu} u^\mu \approx e^{-\nu/2} \left[e^{\lambda/2} v^r, r v^\theta, r \sin\theta v^\varphi \right] \quad (65)$$

is the physical velocity of the fluid in the locally flat coordinate system \tilde{x}^μ . For superfluid matter Eq. (64) should be modified, because in this case not only motion of the normal liquid component contribute to ϵ_{kin} but also that of the superfluid component. Using formula (56) of Kantor & Gusakov (2009) one obtains⁵

$$\epsilon_{\text{kin}} = \frac{1}{2} \left\{ (P + \varepsilon) [\tilde{u}^j]^2 + Y_{ik} \left[\mu_i \tilde{w}_{(k)}^j \tilde{u}_j + \mu_k \tilde{w}_{(i)}^j \tilde{u}_j + \tilde{w}_{(i)}^j \tilde{w}_{(k)j} \right] \right\}, \quad (66)$$

where $\tilde{w}_{(i)}^j = [\partial \tilde{x}^j / \partial x^\mu] w_{(i)}^\mu$. Taking into account Eqs. (3)–(5) and (35), Eq. (66) can be rewritten as

$$\epsilon_{\text{kin}} = \frac{1}{2} (P + \varepsilon) \left\{ \left[\tilde{U}_{(b)}^j \right]^2 + y \left[\tilde{X}^j \right]^2 \right\}, \quad (67)$$

where we neglected ‘temperature’ term TS in the expression (35). In Eq. (67)

$$y \equiv \frac{n_b Y_{\text{pp}}}{\mu_n (Y_{\text{nn}} Y_{\text{pp}} - Y_{\text{np}}^2)} - 1, \quad (68)$$

$$\tilde{U}_{(b)}^j = \frac{\partial \tilde{x}^j}{\partial x^\mu} U_{(b)}^\mu = e^{-\nu/2} \left[e^{\lambda/2} v_{(b)}^r, r v_{(b)}^\theta, r \sin\theta v_{(b)}^\varphi \right], \quad (69)$$

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^\mu} X^\mu = \left[e^{\lambda/2} X^r, r X^\theta, r \sin\theta X^\varphi \right]. \quad (70)$$

Now, using Eqs. (47), (48), (51), and (52) let us express (69) and (70) through the functions $W_b(r)$, $V_b(r)$, $W_{\text{sfl}}(r)$, and $V_{\text{sfl}}(r)$, and then substitute Eq. (67) for ϵ_{kin} into (63). After integrating Eq. (63) over $\sin\theta d\theta d\varphi$ (in the same way as it was done in Thorne & Campolattaro 1967) and making use of Eq. (62), one arrives at the following expression for E_{mech} ,

$$E_{\text{mech}}(t) = E_{\text{mech}(b)}(t) + E_{\text{mech(sfl)}}(t), \quad (71)$$

where we tentatively presented E_{mech} as a sum of two terms related to the baryon motion as a whole $E_{\text{mech}(b)}$ and an additional term $E_{\text{mech(sfl)}}$ appearing because of the superfluid motion,

$$E_{\text{mech}(b)}(t) = \frac{1}{2} \sigma^2 e^{-2t/\tau} \int_0^R (P + \varepsilon) e^{(\lambda-\nu)/2} r^{2l} \left[W_b^2 + l(l+1) V_b^2 \right] dr, \quad (72)$$

$$E_{\text{mech(sfl)}}(t) = \frac{1}{2} \sigma^2 e^{-2t/\tau} \int_0^R (P + \varepsilon) e^{(\lambda-\nu)/2} r^{2l} y \left[W_{\text{sfl}}^2 + l(l+1) V_{\text{sfl}}^2 \right] dr. \quad (73)$$

Strictly speaking, the functions $W_b(r)$, $V_b(r)$, $W_{\text{sfl}}(r)$, and $V_{\text{sfl}}(r)$ in these formulas are complex, that is, instead of, for example, $W_b(r)^2$ one should write $|W_b(r)|^2$. Notice, however, that all these functions [as well as $H_0(r)$, $H_1(r)$, $H_2(r)$, and $K(r)$] are defined up to the same arbitrary complex multiplicative constant. Since $\sigma \gg 1/\tau$ (dissipation is weak), one can always choose the constant in such a way, that the real parts of *all* these functions would be much greater than their imaginary parts (e.g.,

⁵ This expression is analogous to the corresponding formula for the kinetic energy density of a nonrelativistic superfluid mixture, obtained by Andreev & Bashkin (1975), see their equation (7).

$\text{Re}[H_2(r)] \gg \text{Im}[H_2(r)]$), so that one could neglect their ‘complexity’. From here on, unless otherwise stated, by the functions $W_b(r)$, $V_b(r)$, $W_{\text{sff}}(r)$, $V_{\text{sff}}(r)$, $H_0(r)$, $H_1(r)$, $H_2(r)$, and $K(r)$ we mean their real parts.

In the absence of superfluidity $W_{\text{sff}} = V_{\text{sff}} = 0$, $W_b = W$, and $V_b = V$. In that case Eq. (72) gives a mechanical energy of a nonsuperfluid star that coincides, up to notations, with the corresponding expression (29) of Thorne & Campolattaro (1967).

4.2 Dissipation rates

The damping time τ_{grav} due to radiation of gravitational waves can be obtained from the equations, describing linear oscillations of NSs (see Sec. 5 below). The goal of the present section is to determine the dissipation rate of oscillation energy due to the bulk $\mathfrak{W}_{\text{bulk}}$ and shear $\mathfrak{W}_{\text{shear}}$ viscosities and, as a consequence, the damping times τ_{bulk} and τ_{shear} .

For that, we turn to the entropy generation equation (25). Using it, one can find rate of change of the (averaged over the oscillation period) thermal energy of a star dE_{th}/dt due to bulk and shear viscosities. Following the derivation of Eq. (34) in Gusakov, Yakovlev & Gnedin (2005), one obtains

$$\frac{dE_{\text{th}}}{dt} = \int_{\text{star}} (\bar{Q}_{\text{bulk}} + \bar{Q}_{\text{shear}}) e^\nu dV, \quad (74)$$

where \bar{Q}_{bulk} and \bar{Q}_{shear} are the values of Q_{bulk} and Q_{shear} , averaged over the oscillation period $2\pi/\sigma$ [see Eqs. (23) and (24)].

Obviously, the increase in the thermal energy E_{th} is accompanied by the decrease of the oscillation energy E_{mech} , that is

$$\mathfrak{W}_{\text{bulk}} = \int_{\text{star}} \bar{Q}_{\text{bulk}} e^\nu dV, \quad (75)$$

$$\mathfrak{W}_{\text{shear}} = \int_{\text{star}} \bar{Q}_{\text{shear}} e^\nu dV. \quad (76)$$

Using these equations, as well as the formulas (23), (24), (58), (59), and the definitions of Sec. 3.2, one gets, after rather lengthy calculations,

$$\frac{1}{\tau_{\text{bulk}}} = \frac{\sigma^2}{4E_{\text{mech}}(0)} \int_0^R r^{2(l+1)} e^{\lambda/2} \left[\sqrt{\xi_2} \beta_1 + \sqrt{\xi_{3n}} \beta_2 \right]^2 dr, \quad (77)$$

$$\begin{aligned} \frac{1}{\tau_{\text{shear}}} &= \frac{\sigma^2}{2E_{\text{mech}}(0)} \int_0^R \eta r^{2(l-1)} e^{\lambda/2} \\ &\times \left\{ \frac{3}{2} (\alpha_1)^2 + 2l(l+1) (\alpha_2)^2 + l(l+1) \left[\frac{1}{2} l(l+1) - 1 \right] V^2 \right\} dr, \end{aligned} \quad (78)$$

where

$$\beta_1(r) = K + \frac{1}{2} H_2 - \frac{1}{r} e^{-\lambda/2} \left[\frac{dW}{dr} + \frac{1}{r} (l+1) W \right] - l(l+1) \frac{V}{r^2}, \quad (79)$$

$$\beta_2(r) = -\frac{1}{r} e^{-\lambda/2} \left[\frac{d(n_b W_{\text{sff}})}{dr} + \frac{1}{r} (l+1) n_b W_{\text{sff}} \right] - l(l+1) \frac{n_b V_{\text{sff}}}{r^2}, \quad (80)$$

$$\alpha_1(r) = \frac{r^2}{3} \left\{ \frac{2}{r} e^{-\lambda/2} \left[\frac{dW}{dr} + (l-2) \frac{W}{r} \right] + K - H_2 - l(l+1) \frac{V}{r^2} \right\}, \quad (81)$$

$$\alpha_2(r) = \frac{r}{2} \left[\frac{dV}{dr} + (l-2) \frac{V}{r} - e^{\lambda/2} \frac{W}{r} \right] e^{-\lambda/2}. \quad (82)$$

As for the mechanical energy (71), to obtain from these formulas τ_{bulk} and τ_{shear} for a nonsuperfluid star, one has to put $W_{\text{sff}} = V_{\text{sff}} = 0$. In that case our Eqs. (77) and (78) should coincide with the corresponding formulas (5) and (6) of Cutler et al. (1990). Unfortunately, direct comparison of these formulas reveals, that our τ_{bulk} and τ_{shear} appear to be 2 times larger. Using, as tests examples, damping of: (i) NS radial oscillations, (ii) p -modes in the NS envelopes, and (iii) sound waves in the nonsuperfluid matter of NSs we checked, that our results reproduce those of Gusakov et al. (2005); Chugunov & Yakovlev (2005); Kantor & Gusakov (2009), obtained in a quite a different way.

5 OSCILLATION EQUATIONS

In order to calculate τ_{bulk} and τ_{shear} one has to determine the oscillation eigenfrequencies σ and eigenfunctions H_0 , H_1 , H_2 , K , W_b , V_b , W_{sff} , and V_{sff} . To do that one needs to formulate oscillation equations. Since the dissipation is weak, when deriving the oscillation equations one can neglect the dissipative terms in the superfluid hydrodynamics of Sec. 2 and put $\tau^{\mu\nu} = 0$ and $\varkappa_n = 0$.

As it was shown in Gusakov & Kantor (2011), equations, describing small linear oscillations of a NS include:

(i) Continuity equations for baryons (8) and electrons (9), that can be written in terms of the baryon and electron number density perturbations, δn_b and δn_e , as

$$\delta n_b = \frac{i}{\omega e^{-\nu/2}} \left[\partial_j (n_b) U_{(b)}^j + n_b U_{(b);\mu}^\mu \right], \quad (83)$$

$$\delta n_e = \delta n_{e(\text{norm})} + \delta n_{e(\text{sfl})}, \quad (84)$$

where j is the spatial index and we defined

$$\delta n_{e(\text{norm})} \equiv \frac{i}{\omega e^{-\nu/2}} \left[\partial_j (n_e) U_{(b)}^j + n_e U_{(b);\mu}^\mu \right], \quad (85)$$

$$\delta n_{e(\text{sfl})} \equiv -\frac{i}{\omega e^{-\nu/2}} \left[\partial_j (n_e) X^j + n_e X_{;\mu}^\mu \right]. \quad (86)$$

(ii) Einstein equations, which can schematically be presented as

$$\delta(R^{\mu\nu} - 1/2 g^{\mu\nu} R) = 8\pi G \delta T^{\mu\nu}, \quad (87)$$

where the perturbation $\delta T^{\mu\nu}$ of the energy-momentum tensor (11) can be expressed in terms of the perturbations of baryon four-velocity $\delta U_{(b)}^\mu$, metric $\delta g_{\mu\nu}$, pressure δP and energy density $\delta \varepsilon$ as

$$\delta T^{\mu\nu} = (\delta P + \delta \varepsilon) U_{(b)}^\mu U_{(b)}^\nu + (P + \varepsilon) \left[U_{(b)}^\mu \delta U_{(b)}^\nu + U_{(b)}^\nu \delta U_{(b)}^\mu \right] + \delta P g^{\mu\nu} + P \delta g^{\mu\nu}. \quad (88)$$

In Eq. (87) $R^{\mu\nu}$ and R are the Ricci tensor and scalar curvature, respectively; G is the gravitation constant.

(iii) ‘Superfluid’ equation, that can be derived from Eqs. (10) and (13) of Sec. 2 (here we present only the spatial components j of this equation) ⁶

$$i \omega (\mu_n Y_{nk} w_{(k)j} - n_b w_{(n)j}) = n_e \partial_j (e^{\nu/2} \delta \mu). \quad (89)$$

Expressing the vectors $w_{(i)}^j$ through X^j in this equation [see Eqs. (3) and (4)], and introducing the redshifted imbalance of chemical potentials $\delta \mu^\infty \equiv e^{\nu/2} \delta \mu$, one can rewrite Eq. (89) as

$$X_j = \frac{i n_e}{\mu_n n_b \omega y} \partial_j (\delta \mu^\infty), \quad (90)$$

where y is defined by Eq. (68). Notice, that this equation dictates the most general form of the superfluid Lagrangian displacement $\xi_{(\text{sfl})}^j$, that was already obtained in Eq. (52) from the symmetry arguments.

Eqs. (83)–(90) should be supplemented with the expressions for the perturbations δP , $\delta \mu$, and $\delta \varepsilon$. To derive them, let us notice that any thermodynamic quantity (e.g., P) in the superfluid matter can be presented as a function of n_b , n_e , T , and $w_{(i)\mu} w_{(k)}^\mu$ (see, e.g., Gusakov 2007). In strongly degenerate matter the dependence of P , $\delta \mu$, and ε on T can be neglected (see, e.g., Reisenegger 1995; Gusakov et al. 2005), while the scalars $w_{(i)\mu} w_{(k)}^\mu$ are quadratically small in a slightly perturbed star [see Sec. 3.2]. Thus, $P = P(n_b, n_e)$, $\delta \mu = \delta \mu(n_b, n_e)$, and $\varepsilon = \varepsilon(n_b, n_e)$. Expanding these functions into Taylor series near the equilibrium, one obtains

$$\delta P = n_b \frac{\partial P}{\partial n_b} \left[\frac{\delta n_b}{n_b} + \tilde{s} \frac{\delta n_{e(\text{norm})}}{n_e} + s \frac{\delta n_{e(\text{sfl})}}{n_e} \right], \quad (91)$$

$$\delta \mu = n_e \frac{\partial \delta \mu}{\partial n_e} \left[z \frac{\delta n_b}{n_b} + \frac{\delta n_{e(\text{norm})}}{n_e} + \frac{\delta n_{e(\text{sfl})}}{n_e} \right], \quad (92)$$

$$\delta \varepsilon = \mu_n \delta n_b. \quad (93)$$

where we made use of Eq. (84), and introduced dimensionless *coupling parameter* s and the quantities \tilde{s} and z ,

$$s \equiv \frac{n_e}{n_b} \frac{(\partial P / \partial n_e)}{(\partial P / \partial n_b)}, \quad (94)$$

$$\tilde{s} \equiv \frac{n_e}{n_b} \frac{(\partial P / \partial n_e)}{(\partial P / \partial n_b)}, \quad (95)$$

$$z \equiv \frac{n_b}{n_e} \frac{(\partial \delta \mu / \partial n_b)}{(\partial \delta \mu / \partial n_e)}. \quad (96)$$

Notice that the variable \tilde{s} is equal to s here. The reason for discriminating between \tilde{s} and s is purely technical: To solve oscillation equations (see Secs. 6 and 7) it turns out to be convenient to develop a perturbation theory in (small) parameter s , at the same time treating the terms depending on \tilde{s} in a *non-perturbative* way (see Sec. 6.2, and, in particular, footnote ⁹

⁶ It is worth to make a number of comments on Eq. (89): (i) In Gusakov & Kantor (2011) this equation was derived under the assumption that the only superfluid species are neutrons (that is $Y_{pi} = 0$). A generalization of this equation to the case of possible proton superfluidity is presented in Chugunov & Gusakov (2011) [see their Eq. (3)]; (ii) In both papers, Gusakov & Kantor (2011); Chugunov & Gusakov (2011), this equation is written with the same mistake. In particular, in Chugunov & Gusakov (2011) one should write $n_e \partial_j (e^{\nu/2} \delta \mu)$ instead of $n_e e^{\nu/2} \partial_j (\delta \mu)$ in the right-hand side of Eq. (3).

there). When deriving Eq. (93) we took into account that $[\partial\varepsilon(n_b, n_e)/\partial n_e] \delta n_e = -\delta\mu \delta n_e$ is a quadratically small quantity, because $\delta\mu = 0$ in equilibrium⁷.

The vector superfluid equation (90) can be substantially simplified, and reduced to a scalar one. For that let us notice that, without any loss of generality, the scalar $\delta\mu^\infty$ can be presented as

$$\delta\mu^\infty = \delta\mu_l(r) Y_l^0(\theta). \quad (97)$$

Employing now Eqs. (90) and (92), one arrives at

$$\delta\mu_l'' + \left(\frac{h'}{h} - \frac{\lambda'}{2} + \frac{2}{r} \right) \delta\mu_l' - e^\lambda \left[\frac{l(l+1)}{r^2} + e^{-\nu/2} \frac{\omega^2}{h\mathfrak{B}} \right] \delta\mu_l = -\frac{\omega^2 e^{\lambda-\nu/2}}{h\mathfrak{B}} \delta\mu_{\text{norm } l}. \quad (98)$$

Here $h = e^{\nu/2} n_e^2 / (\mu_n n_b y)$, $\mathfrak{B} \equiv \partial\delta\mu(n_b, n_e)/\partial n_e$, and prime (\prime) means derivative with respect to the radial coordinate r . Furthermore, $\delta\mu_{\text{norm } l}(r)$ in Eq. (98) is defined by

$$\delta\mu_{\text{norm}}^\infty = \delta\mu_{\text{norm } l}(r) Y_l^0(\theta), \quad (99)$$

where

$$\delta\mu_{\text{norm}}^\infty \equiv e^{\nu/2} n_e \mathfrak{B} \left[z \frac{\delta n_b}{n_b} + \frac{\delta n_e (\text{norm})}{n_e} \right] \quad (100)$$

is a part of $\delta\mu^\infty$, which depends on $\delta g^{\mu\nu}$ and $U_{(b)}^\mu$ and is independent of the superfluid degrees of freedom X^j [see Eqs. (83) and (85)]. The function $\delta\mu_{\text{norm } l}(r)$ can be easily rewritten in terms of $H_0(r)$, $H_1(r)$, $H_2(r)$, $K(r)$, $W_b(r)$, and $V_b(r)$ with the help of Eqs. (37), (43), (44), (47), (51), (83), and (85). One obtains

$$\delta\mu_{\text{norm } l} = e^{\nu/2} n_b \frac{\partial\delta\mu(n_b, x_e)}{\partial n_b} r^l \beta_1, \quad (101)$$

where $x_e \equiv n_e/n_b$ and $\beta_1(r)$ is given by Eq. (79) with W_b and V_b instead of, respectively, W and V .

Finally, let us mention one important property, that follows from the oscillation equations and quasineutrality condition (3). If neutrons in a nonrotating nonmagnetized star are normal (i.e. $Y_{nn} = Y_{np} = 0$), while protons are superfluid ($Y_{pp} \neq 0$), then oscillation eigenfrequencies and eigenfunctions for such star will be indistinguishable from that for a normal star of the same mass (where both protons and neutrons are nonsuperfluid).

6 OUR APPROACH

6.1 Decoupling of superfluid and normal modes

In principle, Eqs. (83)–(101) allow one to study the nonradial oscillations of superfluid NSs and thus to determine the spectrum of eigenfrequencies ω , eigenfunctions H_0 , H_1 , ..., W_{sfl} , and V_{sfl} , and hence the damping times τ_{grav} , τ_{bulk} , and τ_{shear} . However, this task can be significantly simplified, if one notes that the dimensionless coupling parameter s (94) is small for realistic equations of state of superdense matter (Gusakov & Kantor 2011). For example, for the equation of state APR (Akmal, Pandharipande & Ravenhall 1998) employed below $s \sim 0.01 \div 0.05$. This means that one can look for the solution to the system of Eqs. (83)–(101) in the form of a series in s . Since s is small, the approximation $s = 0$ is already quite accurate. Indeed, as it was shown in Gusakov & Kantor (2011) with the example of radial oscillations, the eigenfrequencies calculated in this approximation differ from the exact ones, on average, by $\sim 1.5 \div 2\%$. Thus, in what follows all calculations are performed assuming $s = 0$.

How this approach simplifies the problem? As it was first demonstrated in Gusakov & Kantor (2011), in the $s = 0$ approximation superfluid degrees of freedom (vectors X^j) completely decouple from the ‘normal’ degrees of freedom [metric perturbations $\delta g_{\mu\nu}$ and baryon four-velocities $\delta U_{(b)}^\mu$]. That is, one has *two* distinct classes of oscillations: ‘superfluid’ and ‘normal’ modes, which are described by independent equations. For superfluid-type oscillations the metric and baryon velocity are not perturbed [$\delta g_{\mu\nu} = 0$ and $\delta U_{(b)}^\mu = 0$], hence these modes *do not emit* gravitational waves; moreover, they are *entirely* localized in the SFL-region. At the same time, the frequencies of normal modes are *indistinguishable* from those of a normal (nonsuperfluid) star of the same mass⁸. Below we discuss in more detail decoupling of superfluid and normal oscillation modes and how this property can be used to calculate the characteristic damping times.

⁷ The equality $\partial\varepsilon(n_b, n_e)/\partial n_e = -\delta\mu$ follows from the second law of thermodynamics (15), which can be rewritten in our case as $d\varepsilon = \mu_n dn_b - \delta\mu dn_e$.

⁸ Here and below by ‘normal modes’ we mean oscillation modes (of approximate solution), that also exist in the normal (nonsuperfluid) star.

6.2 A strategy to calculate the damping times

So, let us formally assume that $s = 0$ (while \tilde{s} is given by Eq. (95) and is non-zero). Then, as follows from Eq. (91), δP equals

$$\delta P = n_b \frac{\partial P}{\partial n_b} \left[\frac{\delta n_b}{n_b} + \tilde{s} \frac{\delta n_e(\text{norm})}{n_e} \right] \quad (102)$$

and is independent of the superfluid degrees of freedom X^μ [see Eqs. (83) and (85)]. Other terms in the expression (88) for $\delta T^{\mu\nu}$ also do not depend on X^μ [in particular, $\delta\varepsilon = \mu_n \delta n_b$ does not depend on X^μ due to Eqs. (83) and (93)]. Thus, we come to conclusion that the linearized Einstein equations (87) depend *only* on perturbations of the metric $g_{\mu\nu}$ and the baryon four-velocity $U_{(b)}^\mu$ and are independent of X^μ . Moreover, it is easy to see, that in the case $s = 0$ these equations (and the corresponding boundary conditions) have exactly the same form as in the absence of superfluidity⁹. Correspondingly, two alternatives are possible when solving the system of Eqs. (83)–(101) in the approximation $s = 0$:

(1) A star oscillates at a frequency which is *not* an eigenfrequency of the Einstein equations (87). In that case, to satisfy Eq. (87), one has to demand

$$H_0 = H_1 = H_2 = K = W_b = V_b = 0. \quad (103)$$

From Eq. (101) it follows then, that $\delta\mu_{\text{norm}l} = 0$ and the superfluid equation (98) decouples from the Einstein equations. As a result we arrive at the ‘source-free’ equation (with the right-hand side vanished), first derived in Chugunov & Gusakov (2011)¹⁰,

$$\delta\mu_l'' + \left(\frac{h'}{h} - \frac{\lambda'}{2} + \frac{2}{r} \right) \delta\mu_l' - e^\lambda \left[\frac{l(l+1)}{r^2} + e^{-\nu/2} \frac{\omega^2}{h\mathfrak{B}} \right] \delta\mu_l = 0. \quad (104)$$

This equation describes *superfluid modes* and should be solved in the stellar region where neutrons are superfluid (SFL-region). It should be supplemented with a number of boundary conditions, discussed in Chugunov & Gusakov (2011) [similar, but more general boundary conditions for Eq. (98) are presented in the Appendix]. Having solved Eq. (104) for $\delta\mu_l$, it is easy to determine the functions W_{sfl} and V_{sfl} using Eqs. (48), (52), and (90). Using W_{sfl} and V_{sfl} , one can find the functions W and V from Eqs. (53), (54), and (103),

$$W = -W_{\text{sfl}}, \quad V = -V_{\text{sfl}}. \quad (105)$$

This information is sufficient to calculate τ_{bulk} and τ_{shear} from Eqs. (77) and (78) [as follows from Eq. (103), $\tau_{\text{grav}} = \infty$ for superfluid modes in the $s = 0$ approximation].

(2) A star oscillates at a frequency which is an eigenfrequency of Einstein equations (87). In that case, the eigenfrequency and eigenfunctions H_0, H_1, H_2, K, W_b , and V_b are *indistinguishable* from the corresponding eigenfrequency and eigenfunctions for an oscillating nonsuperfluid NS [we recall, that for the nonsuperfluid star $W_b = W, V_b = V$, because $W_{\text{sfl}} = V_{\text{sfl}} = 0$, see Eqs. (53) and (54)]. There is, however, one very important difference: for a superfluid star the functions W_{sfl} and V_{sfl} *do not vanish* in the SFL-region and are comparable there to W_b and V_b . As follows from Eqs. (77) and (78), the damping times τ_{bulk} and τ_{shear} depend on these functions [as well as on $W = W_b - W_{\text{sfl}}$ and $V = V_b - V_{\text{sfl}}$], that is why the determination of W_{sfl} and V_{sfl} is a necessary task.

To determine these functions we make use of Eq. (98). Since the oscillation frequency $\omega = \sigma + i/\tau_{\text{grav}}$ and the eigenfunctions H_0, H_1, H_2, K, W_b , and V_b are already known, we can, using Eq. (101), calculate $\delta\mu_{\text{norm}l}$ and determine a ‘source’ in the right-hand side of Eq. (98). This source plays a role of an external driving force, that makes the superfluid equation (98) ‘oscillate’ at the frequency ω , which is *not* an eigenfrequency for this equation¹¹. As a result, the function $\delta\mu_l(r)$ will be nonzero. To determine it one has to specify the boundary conditions for Eq. (98); they are formulated in Appendix. Having solved Eq. (98) numerically and having defined $\delta\mu_l(r)$, one can calculate the functions W_{sfl} and V_{sfl} , using Eqs. (48), (52), (90), and (97).

Summarizing, in the approximation $s = 0$ the eigenfrequencies and eigenfunctions H_0, H_1, H_2, K, W_b , and V_b , (and hence τ_{grav}) for the normal modes appear to be *the same* as for a nonsuperfluid star. At the same time the eigenfunctions W_{sfl} and V_{sfl} are non-zero in the SFL-region and should be determined from Eq. (98). As a result, the damping times τ_{bulk} and

⁹ This is the main advantage of treating \tilde{s} in a non-perturbative way. Notice, however, that this trick leads to somewhat ‘excessive’ accuracy of the approximate solution to oscillation equations: the retained terms depending on \tilde{s} may lead to smaller correction to the solution than the s -dependent terms which were ignored. Bearing this in mind and with the aim to simplify consideration, in Gusakov & Kantor (2011) it was assumed that both parameters s and \tilde{s} vanish in the $s=0$ approximation. Such an approach is also possible. In that case, strictly speaking, the resulting Einstein equations would differ slightly from the equations describing oscillations of a nonsuperfluid NSs. In particular, instead of the standard adiabatic index of the ‘frozen’ npe-matter $\gamma_{\text{fr}} = (n_b/P)[\partial P(n_b, x_e)/\partial n_b]$ the new index would appear, $\gamma = (n_b/P)[\partial P(n_b, n_e)/\partial n_b]$. However, this difference is not essential, because s and \tilde{s} are small.

¹⁰ The corresponding equation (5) of Chugunov & Gusakov (2011), contains a mistake, that was corrected in the second version of the manuscript in arXiv (see arXiv:1107.4242v2).

¹¹ In the present paper, in all numerical calculations we used σ instead of ω in Eq. (98), because $\sigma \gg 1/\tau_{\text{grav}}$. Also, when calculating $\delta\mu_{\text{norm}l}$ we only employed the real parts of eigenfunctions H_0, H_1, H_2, K, W_b , and V_b [see a note after Eq. (73)].

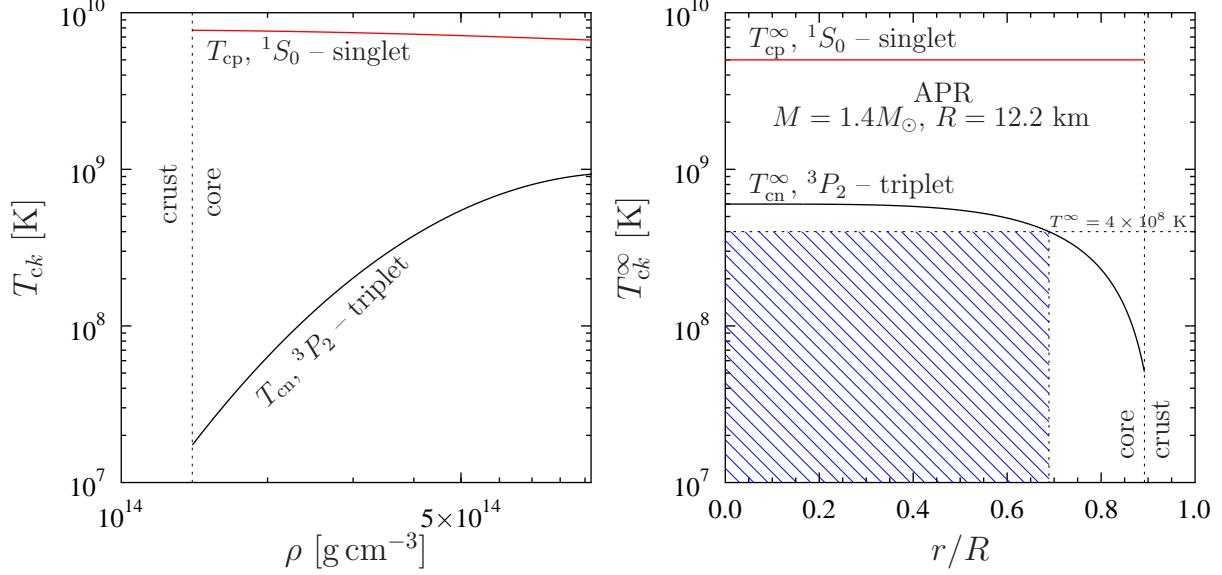


Figure 1. (color online) Left panel: Nucleon critical temperatures T_{ck} ($k = n, p$) versus density ρ for model 1. Right panel: Redshifted critical temperatures T_{ck}^{∞} versus radial coordinate r (in units of R) for model 1.

τ_{shear} , defined by Eqs. (77) and (78), will *differ* from the corresponding times, calculated using the ordinary (nonsuperfluid) hydrodynamics (even if one takes into account the effects of superfluidity on the kinetic coefficients).

7 RESULTS

Let us apply the approach, suggested in the previous section, to determine the frequency spectrum and damping times for an oscillating superfluid NS. But first let us discuss its equilibrium model.

7.1 Microphysics input and equilibrium model

As mentioned in Sec. 2, we consider the simplest npe-composition of NS core. We adopt APR equation of state (Akmal et al. 1998) parametrized by Heiselberg & Hjorth-Jensen (1999) in the core and the equation of state by Negele & Vautherin (1973) in the crust.

All numerical results presented here are obtained for a NS with the mass $M = 1.4M_{\odot}$. The circumferential radius for such star is $R = 12.2$ km, the central density is $\rho_c = 9.26 \times 10^{14}$ g cm $^{-3}$. The crust-core interface lies at the distance $R_{cc} = 10.9$ km from the centre.

When modeling the effects of superfluidity we assume the triplet pairing of neutrons and singlet pairing of protons in the NS core. The neutron superfluidity in the stellar crust is neglected; it should not affect strongly the global oscillations of NSs.

We consider two models of nucleon superfluidity: model ‘1’ (simplified) and model ‘2’ (more realistic). In the model 1 the redshifted proton critical temperature is constant over the core, $T_{cp}^{\infty} \equiv T_{cp} e^{\nu/2} = 5 \times 10^9$ K; the redshifted neutron critical temperature $T_{cn}^{\infty} \equiv T_{cn} e^{\nu/2}$ increases with the density ρ and reaches the maximum value $T_{cn \text{ max}}^{\infty} = 6 \times 10^8$ K at the stellar centre ($r=0$). This model corresponds to the model 3 of Kantor & Gusakov (2011).

In the model 2 both critical temperatures T_{cn} and T_{cp} are density dependent. This model does not contradict the results of microscopic calculations (see, e.g., Lombardo & Schulze 2001; Yakovlev et al. 1999) and is similar to the nucleon pairing models used to explain observations of the cooling NS in Cassiopea A supernova remnant (Shternin et al. 2011).

The models 1 and 2 are shown in Figs. 1 and 2, respectively¹². The function $T_{ci}(\rho)$ in both figures is shown in the left panels, while the right panels demonstrate the dependence $T_{ci}^{\infty}(r)$ [$i = n$ and p]. With the decrease of the redshifted temperature T^{∞} the size of the SFL-region [given by the condition $T < T_{cn}(r)$, or, equivalently, $T^{\infty} < T_{cn}^{\infty}(r)$] increases or remains unchanged. For instance, the SFL-region corresponding to $T^{\infty} = 4 \times 10^8$ K, is shaded in Figs. 1 and 2. One can see that for the model 2 there can be three-layer configurations of a star with no neutron superfluidity in the centre and in

¹² Fig. 2 is a slightly modified version of Fig. 1 from Chugunov & Gusakov (2011).

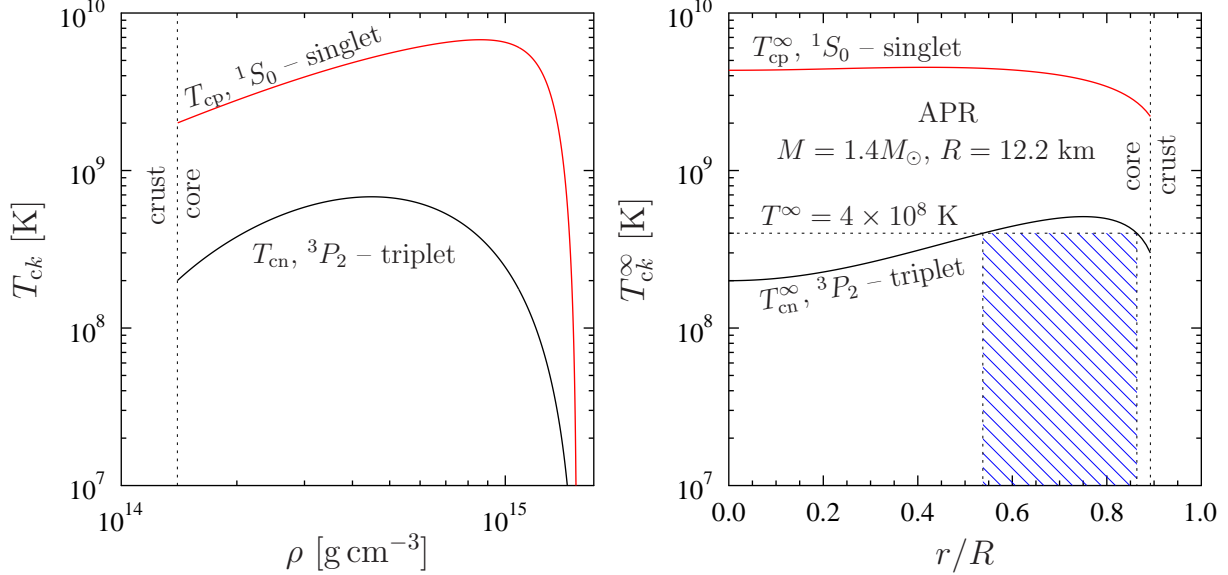


Figure 2. (color online) The same as in Fig. 1 but for model 2.

the outer region but with superfluid intermediate region. On the contrary, in the model 1 only two-layer configurations are possible.

The entrainment matrix Y_{ik} is calculated for the superfluidity models 1 and 2 in a way similar to how it was done in Kantor & Gusakov (2011).

When analyzing viscous dissipation in oscillating NSs we allow for the damping due to shear and bulk viscosities. For the shear viscosity coefficient η we take the electron shear viscosity η_e , calculated in Shternin & Yakovlev (2008). We neglect the nucleon shear viscosity because: (i) it is poorly known even for nonsuperfluid matter and (ii) it appears to be less than the electron shear viscosity in the core at $T \ll T_{cp}$ (Shternin & Yakovlev 2008).

The bulk viscosity coefficients are calculated as described by Gusakov (2007); Gusakov & Kantor (2008); Kantor & Gusakov (2011). Since the direct URCA process is closed for our stellar model with $M = 1.4M_{\odot}$, the main contributor to the bulk viscosity is the modified URCA process.

7.2 Oscillations of a nonsuperfluid star

As follows from Sec. 6.2, before considering oscillations of a superfluid NS one should study those of a normal (nonsuperfluid) star of the same mass. To this aim, we have determined the eigenfrequencies and eigenfunctions of the radial and nonradial oscillation modes for a nonsuperfluid NS of mass $M = 1.4M_{\odot}$ and equation of state APR (see Sec. 7.1). We have solved the equations describing radial and nonradial perturbations of a nonrotating star in general relativity. These equations are derived by expanding the perturbed Einstein's equations in tensorial spherical harmonics in an appropriate gauge, and are integrated in the frequency domain.

Stellar modes are defined as solutions of the perturbed equations which are regular at the centre and with vanishing Lagrangian pressure perturbation at the surface, and (if $l > 1$) which behave as a pure outgoing wave at infinity; as discussed above, such solutions have complex frequencies $\omega = \sigma + i/\tau$. If $l \leq 1$, instead, the frequency is real and the mode is not associated to gravitational emission.

The oscillation modes are classified according to the source of the restoring force which prevails in bringing the perturbed element of fluid back to the equilibrium position; for instance, we have a g -mode if the restoring force is mainly provided by buoyancy, a p -mode if it is due to a gradient of pressure, and so on.

The radial modes are calculated as described in Gusakov et al. (2005). To calculate the nonradial modes we follow the formulation of Lindblom & Detweiler (1983) and Detweiler & Lindblom (1985). In their formulation, the equations for nonradial perturbations can be expressed, inside the star, as a system of first-order differential equations in the variables H_0, H_1, H_2, K, W_b , and V_b defined in Sec. 5. Outside the star, they reduce to a simple, second-order differential equation (the Zerilli equation). By numerical integration of these equations (the procedure we have followed is described in detail, e.g., in Burgio et al. 2011) we find, for each value of the multipolarity l , the (complex) eigenfrequencies ω and the corresponding

Table 1. Frequency σ (in units of 10^4 s^{-1} and in units of $\tilde{\sigma} = c/R \approx 2.46 \times 10^4 \text{ s}^{-1}$) and the damping time τ_{grav} (in seconds) for various oscillation modes of a nonsuperfluid NS. The first column shows the multipolarity l of modes and their names.

l , mode	$\sigma/(10^4 \text{ s}^{-1})$	$\sigma/\tilde{\sigma}$	$\tau_{\text{grav}} \text{ (s)}$
0, F	1.703	0.691	∞
0, $1H$	4.080	1.656	∞
0, $2H$	5.732	2.327	∞
1, p_1	2.893	1.175	∞
2, f	1.155	0.469	0.212
2, p_1	3.720	1.510	3.799
3, f	1.554	0.631	18.24
3, p_1	4.360	1.770	33.26

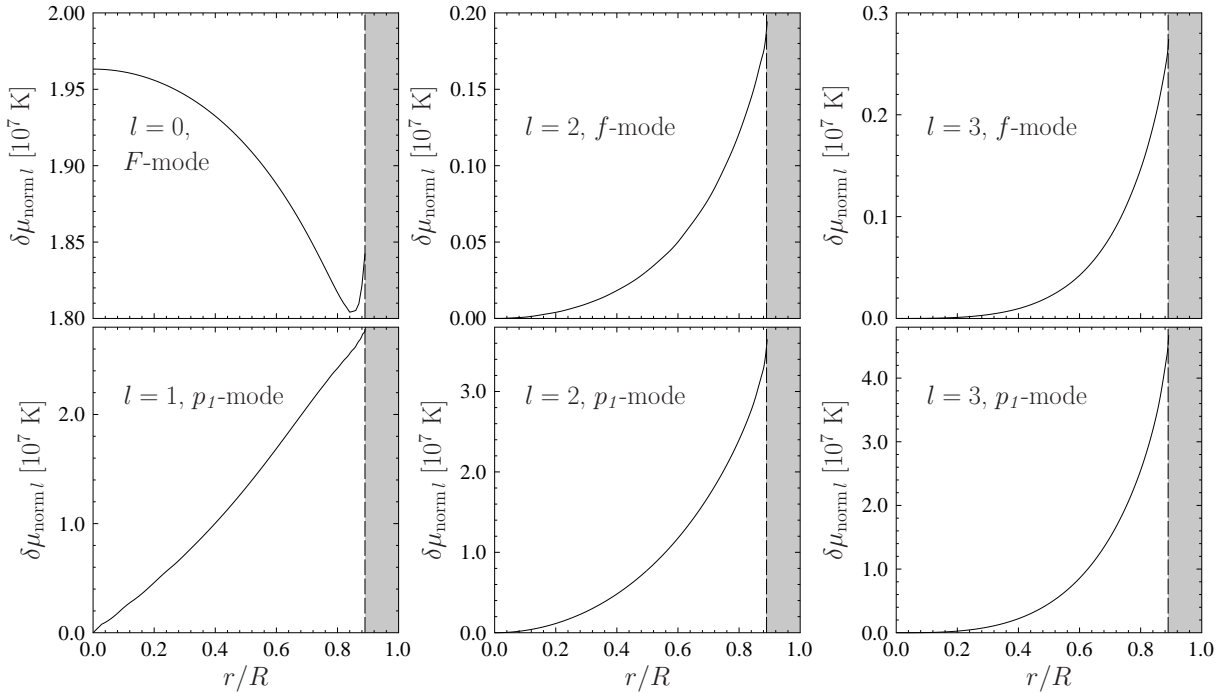


Figure 3. The function $\delta\mu_{\text{norm } l}$ (in units of 10^7 kelvins) versus r for fundamental radial F -mode as well as for p_1 - and f -modes with multiplicities $l = 1, 2$, and 3 (see the footnote ¹³). The energy of each oscillation mode is 10^{43} erg. Shaded region corresponds to crust, where $\delta\mu_{\text{norm } l}$ is not defined and was not plotted.

eigenfunctions $H_0(r)$, $H_1(r)$, $H_2(r)$, $K(r)$, $W_b(r)$, and $V_b(r)$. The results of our computations are summarized in Table 1 and illustrated in Figs. 3 and 4.

Table 1 presents the real parts of the eigenfrequencies $\text{Re}(\omega) = \sigma$ (measured in units of 10^4 s^{-1} and in units of $\tilde{\sigma} \equiv c/R \approx 2.46 \times 10^4 \text{ s}^{-1}$) and the characteristic gravitational damping times τ_{grav} (in seconds) for the modes with $l = 0$ (fundamental F -mode and first two overtones $1H$ and $2H$), $l = 1$ (dipole p_1 -mode), $l = 2$ (quadrupole f - and p_1 -modes), and $l = 3$ (octupole f - and p_1 -modes) ¹³. One can see, that $\sigma \gg 1/\tau_{\text{grav}}$ in all these cases. That is, damping due to emission of gravitational waves occurs on a time scale much longer than the oscillation period.

Using the definition (99) and Eq. (101) we have determined, in terms of the eigenfunctions $H_0(r)$, $H_1(r)$, \dots , $V_b(r)$, the function $\delta\mu_{\text{norm } l}(r)$ and, consequently, the quantity $\delta\mu_{\text{norm}}^\infty(r, \theta) = \delta\mu_{\text{norm } l}(r) Y_l^0(\theta)$ for each mode. As follows from Eqs. (92) and (100), for a *nonsuperfluid* star $\delta\mu_{\text{norm}}^\infty(r, \theta)$ is simply a redshifted imbalance of chemical potentials, $\delta\mu^\infty = \delta\mu_{\text{norm}}^\infty$. The function $\delta\mu_{\text{norm } l}(r)$, entering Eq. (98), is shown in Fig. 3 for the oscillation modes from Table 1. It is normalized such that the mechanical energy of oscillations is 10^{43} erg. The shaded region corresponds to the crust of the star, where $\delta\mu_{\text{norm } l}(r)$ is not defined (protons are bound in nuclei there). As seen in the figure, $|\delta\mu_{\text{norm } l}(r)|$ for f -modes is about one order of magnitude

¹³ The f -mode is absent in case of $l = 1$.

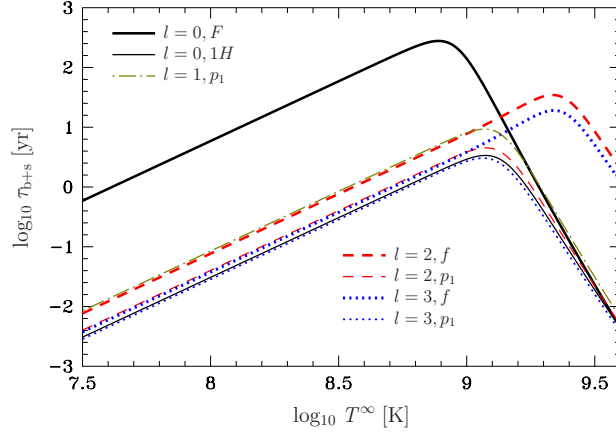


Figure 4. (color online) Damping times $\tau_{b+s} \equiv (\tau_{\text{bulk}}^{-1} + \tau_{\text{shear}}^{-1})^{-1}$ versus T^∞ for various oscillation modes. The effects of superfluidity are partially taken into account, as described in the text. Thick and thin solid lines correspond to radial ($l = 0$) F - and $1H$ -modes, respectively; dot-dashed line – to dipole ($l = 1$) p_1 -mode; thick and thin dashes – to quadrupole ($l = 2$) f - and p_1 -modes, respectively; thick and thin dots – to octupole ($l = 3$) f - and p_1 -modes, respectively.

smaller than for p -modes. This is not surprising, since matter is only weakly compressed during f -mode oscillations, so that a deviation from beta-equilibrium (when $\delta\mu^\infty = 0$) is small. The functions $\delta\mu_{\text{norm } l}(r)$ are employed to calculate the damping times of a superfluid NS in Sec. 7.4.

Fig. 4 shows the viscous damping time $\tau_{b+s} \equiv (\tau_{\text{bulk}}^{-1} + \tau_{\text{shear}}^{-1})^{-1}$ as a function of T^∞ for a set of oscillation modes. The solid lines corresponds to radial ($l = 0$) modes F and $1H$; dot-dashed line to dipole ($l = 1$) mode p_1 ; dashed lines to quadrupole ($l = 2$) modes f and p_1 ; dotted lines to octupole ($l = 3$) modes f and p_1 . To calculate τ_{b+s} we used the formulas for τ_{bulk} and τ_{shear} , applicable for the ordinary hydrodynamics of a nonsuperfluid liquid¹⁴. However, we allow for the effects of superfluidity when calculating the kinetic coefficients η and ξ_2 (the other bulk viscous coefficients do not appear in the normal fluid hydrodynamics). To calculate η and ξ_2 we adopt the nucleon superfluidity model 2 (see Sec. 7.1). Such an approximate approach to accounting for the effects of superfluidity is commonly used in the literature, but it is not fully consistent. The results of a more consistent approach (see Sec. 6) are discussed below in Sec. 7.4.

As follows from Fig. 4, the dependence of τ_{b+s} on T^∞ is a power-law at $T^\infty \lesssim 6 \times 10^8$ K. At such T^∞ the proton superfluidity is ‘strong’ ($T^\infty \ll T_{\text{cp}}^\infty$). In that case the bulk viscosity is exponentially suppressed (Haensel, Levenfish & Yakovlev 2001), while the shear viscosity $\eta \propto 1/(T^\infty)^2$ (Shternin & Yakovlev 2008) and dominates. As a result, $\tau_{b+s} \propto (T^\infty)^2$. At high enough temperatures $T^\infty \gtrsim 6 \times 10^8$ K the damping due to the bulk viscosity starts to prevail; this results in decreasing of τ_{b+s} with growing T^∞ (the curves in Fig. 4 bend down). At such T^∞ the neutrons are normal and the proton superfluidity is weak or absent. Neglecting the proton superfluidity, one obtains $\xi_2 \propto (T^\infty)^6$ (Haensel et al. 2001), hence $\tau_{b+s} \propto 1/(T^\infty)^6$.

Let us note that the curves for f -modes in Fig. 4 (thick dashed line and thick dots) bend down later than others; for them the shear viscosity is the dominant mechanism of damping up to $T^\infty \approx 2.0 \times 10^9$ K. This is not surprising, since, as it was noted above, for f -modes the deviation from beta-equilibrium is small ($\delta\mu^\infty$ is reduced by an order of magnitude in comparison to p -modes, see Fig. 3), hence damping due to the bulk viscosity is suppressed (the relation between $\delta\mu^\infty$ and τ_{bulk} was discussed in detail, e.g., in Gusakov et al. 2005). As a result, τ_{b+s} approaches its ‘bulk viscosity’ asymptote $\tau_{b+s} \propto 1/(T^\infty)^6$ at higher temperatures $T^\infty > 2.0 \times 10^9$ K.

7.3 Frequency spectrum for superfluid NSs

First of all let us consider the frequency spectrum for radial oscillations of a superfluid neutron star employing the simplified model 1 of nucleon superfluidity. For such model this problem was discussed in detail by Kantor & Gusakov (2011), where it was solved exactly. Here we compare this exact solution with the approximate calculations obtained in the $s = 0$ approximation (see Sec. 6). Such a comparison is very useful, since it allows one to make a conclusion about applicability of the approximate approach in the case of nonradial oscillations, where the exact solution is not attempted.

The eigenfrequencies σ of radial pulsations (in units of $\bar{\sigma}$) versus $T_8^\infty = T^\infty/(10^8 \text{ K})$ are shown in Fig. 5(a, b, c). In Fig. 5(a) this dependence was obtained assuming that superfluid and normal modes are completely decoupled ($s = 0$ approximation). The thick solid lines demonstrate the first three normal (nonsuperfluid) radial modes F , $1H$, and $2H$. As one expects, their frequencies do not depend on T^∞ . The dashes are for the first six superfluid modes $1, \dots, 6$, which are the solutions to Eq.

¹⁴ More precisely, we used Eqs. (77) and (78) with $W_{\text{sf}} = V_{\text{sf}} = 0$.

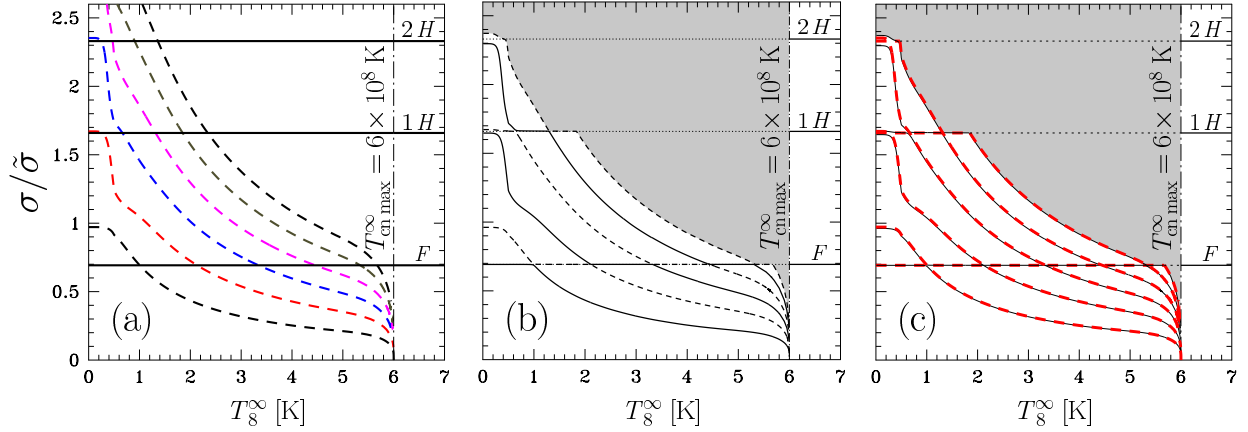


Figure 5. (color online) Eigenfrequencies σ (in units of $\bar{\sigma}$) of radial oscillations versus $T_8^\infty = T^\infty/(10^8 \text{ K})$ for model 1 of nucleon superfluidity. (a) Approximate spectrum. First three normal modes (F , $1H$, and $2H$) are shown by the solid lines; first six superfluid modes $1, \dots, 6$ are shown by dashes. (b) Exact spectrum. Alternate solid and dashed lines show the first six exact modes (I, ..., VI) of a radially oscillating star. No spectrum is plotted in the shaded region. (c) Approximate (dashed lines) and exact (solid lines) spectra. At $T^\infty > T_{\text{cn max}}^\infty = 6 \times 10^8 \text{ K}$ all neutrons are normal and the spectrum is that of a nonsuperfluid star.

(104). These modes, on the contrary, strongly depend on T^∞ and approach their temperature-independent asymptotes only at $T^\infty \lesssim 5 \times 10^7 \text{ K}$ (when the entire NS core is superfluid and Y_{ik} does not depend on T^∞). At $T^\infty > T_{\text{cn max}}^\infty = 6 \times 10^8 \text{ K}$ all neutrons are normal so that superfluid modes do not exist.

Fig. 5(b) demonstrates the results of the exact solution to Eqs. (83)–(101) obtained by Kantor & Gusakov (2011) for radial oscillations of a superfluid neutron star. The frequencies σ of the first six oscillation modes (I, ..., VI) as functions of T_8^∞ are shown by alternate solid and dashed lines. No spectrum is plotted in the gray-shaded area. One can observe that the approximate spectrum [Fig. 5(a)] is very similar to the exact spectrum [Fig. 5(b)]. However, there is one important difference: instead of *crossings* of superfluid and normal modes in Fig. 5(a) we have *avoided crossings* of the modes in Fig. 5(b). At these points the superfluid mode turns into the normal one and vice versa. As it was discussed in details in Gusakov & Kantor (2011); Kantor & Gusakov (2011), this is not surprising, since in a vicinity of avoided crossings the Einstein equations (87) and superfluid equation (98) interact resonantly, so that approximation of completely decoupled superfluid and normal modes ($s = 0$) is inapplicable¹⁵.

For comparison, in Fig. 5(c) we plot both the approximate (dashed lines) and exact (solid lines) spectra. The agreement between both spectra is very good: the difference is less than a few per cent.

Such a close agreement of the exact and approximate results for radial oscillations allows us to analyse the spectrum of nonradial oscillations using the same approximation $s = 0$. The results of this analysis are shown in Fig. 6 for more realistic model 2 of nucleon superfluidity (see Sec. 7.1 and Fig. 2). Superfluid modes shown in this figure have been already studied in detail in our recent paper (Chugunov & Gusakov 2011). Thus, here we discuss them only briefly.

Fig. 6 contains five panels. Four upper panels present eigenfrequencies σ as functions of T_8^∞ for normal modes from Table 1 (thick horizontal lines) and for superfluid modes (dashes) with multipolarities $l = 0, 1, 2$, and 3 . For each l there is an infinite set of superfluid modes whose eigenfunctions $\delta\mu_l$ differ by the number of radial nodes n ; in the figure we plot the first 25 of them. The lower panel demonstrates broadening of the SFL-region with decreasing T_8^∞ (SFL-region is shown by hatches). For model 2 (which we employ here) the redshifted neutron critical temperature $T_{\text{cn}}^\infty(r)$ has a maximum at $T_{\text{cn max}}^\infty \approx 5.1 \times 10^8 \text{ K}$ (right vertical dotted line). The neutron superfluidity reaches the stellar centre at $T^\infty = T_{\text{cn}}^\infty(0) \approx 2 \times 10^8 \text{ K}$ (left vertical dotted line). At $T^\infty > T_{\text{cn max}}^\infty$ all neutrons are normal, hence only normal modes exist in the star. At $T^\infty < T_{\text{cn}}^\infty(0)$ the core is completely occupied by the neutron superfluidity. One can see that the behaviour of superfluid modes differs strongly at $T^\infty > T_{\text{cn}}^\infty(0)$ and at $T^\infty < T_{\text{cn}}^\infty(0)$. This feature was discussed in Chugunov & Gusakov (2011); Kantor & Gusakov (2011), where it was demonstrated that (roughly speaking) the frequencies σ of superfluid modes scale with Y_{nn} and R_{sfl} as $\sigma \sim \sqrt{Y_{\text{nn}}}/R_{\text{sfl}}$, where R_{sfl} is the size of the SFL-region. With the increasing of temperature Y_{nn} decreases, while the size of the SFL-region can either decrease [at $T^\infty > T_{\text{cn}}^\infty(0)$] or remain constant [at $T^\infty < T_{\text{cn}}^\infty(0)$]. As a result, there is a partial compensation of these two tendencies at $T^\infty > T_{\text{cn}}^\infty(0)$ (hence, the frequency changes only weakly), while at $T^\infty < T_{\text{cn}}^\infty(0)$ the effect of decreasing of Y_{nn} is not compensated (hence, σ decreases with growing T^∞). At $T^\infty \lesssim 5 \times 10^7 \text{ K}$ Y_{ik} does not depend on T^∞ , and, as in the case of radial pulsations, the frequencies approach their low-temperature asymptotes.

¹⁵ Thus, it would not be correct to say that any *real* oscillation mode of a superfluid star is either purely superfluid or purely normal: for some T^∞ it can show itself as a superfluid, but for other T^∞ it can behave as a normal mode [see Fig. 5(b)].

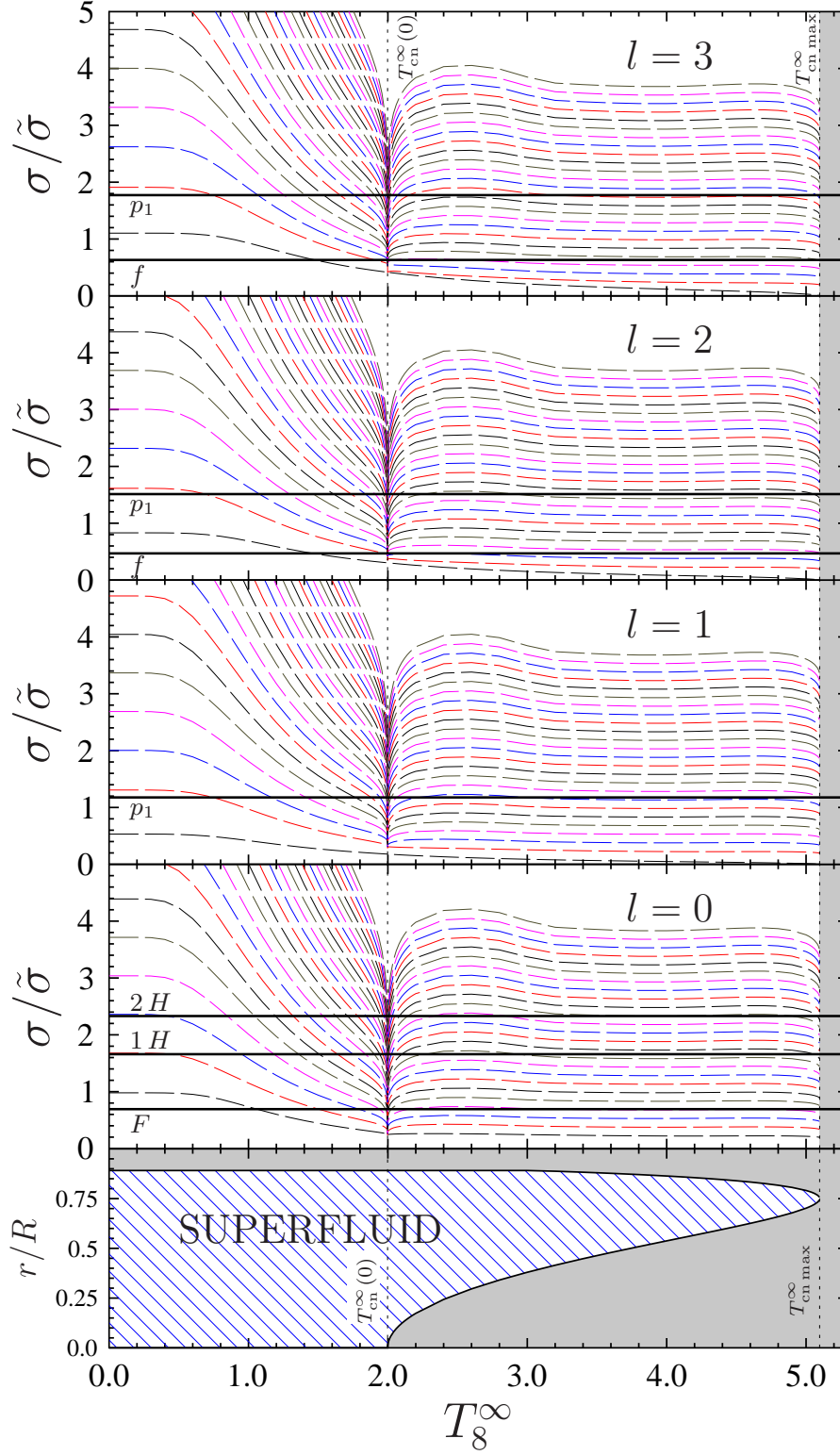


Figure 6. (color online) Eigenfrequencies σ versus T_8^∞ for model 2 of nucleon superfluidity and for multiplicities $l = 0, 1, 2$, and 3 . For each l we plot first few normal modes (solid lines) and first 25 superfluid modes (dashed lines), whose eigenfunctions $\delta\mu_l$ differ by the number of radial nodes n . At $T_8^\infty \leq T_{\text{cn}}^\infty(0) \approx 2 \times 10^8$ K (see the left vertical dotted line), neutron superfluidity occupies the stellar centre. The bottom panel demonstrates the variation of the SFL-region (shown by hatching) with T_8^∞ . Shaded area in all panels shows the region where all neutrons are normal.

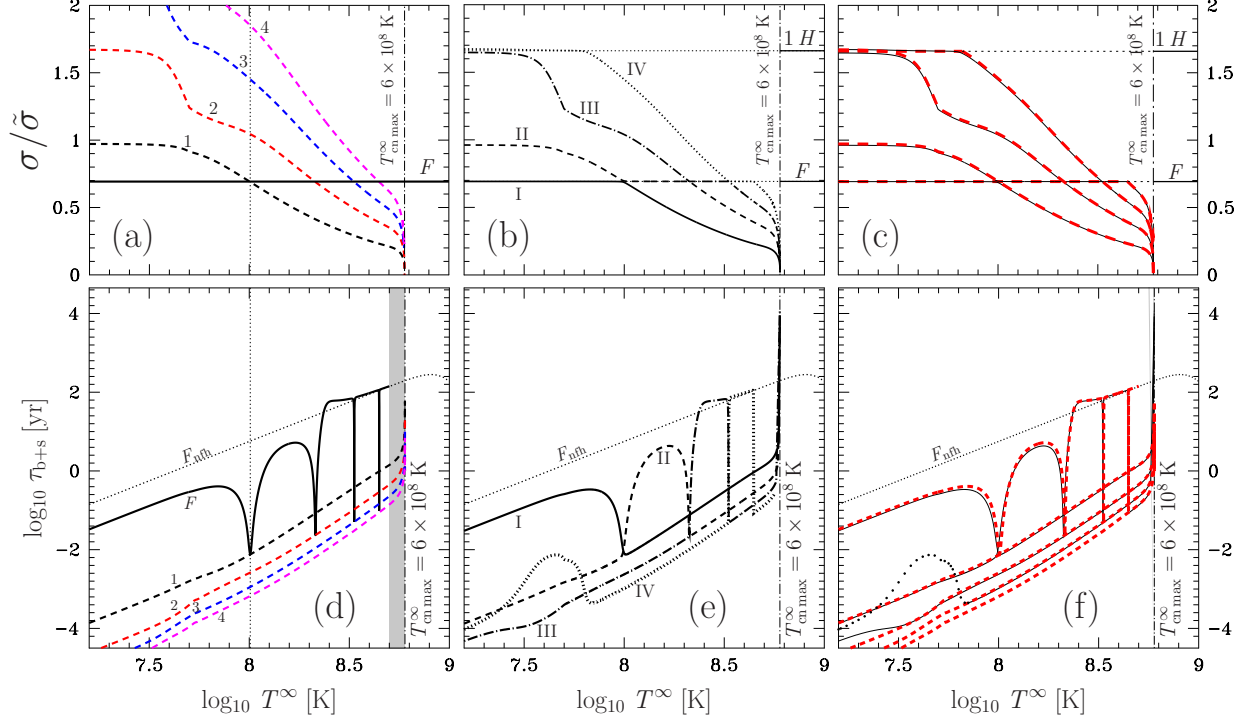


Figure 7. (color online) Eigenfrequencies σ [panels (a, b, c)] and damping times τ_{b+s} [panels (d, e, f)] of a radially oscillating NS versus T^∞ for model 1 of nucleon superfluidity. Panels (a, d): Approximate solution (normal F -mode and first four superfluid modes $1, \dots, 4$ are shown by solid and dashed lines, respectively); Panels (b, e): Exact solution (first four exact modes I, \dots , IV are shown by solid, dashed, dot-dashed, and dotted lines, respectively); Panels (c, f): Both approximate (dashed lines) and exact (solid lines) solutions. Panels (a, b, c) are the same spectra as those plotted, respectively, in Fig. 5(a, b, c). Normal F -mode is not shown in the shaded region because of technical reasons (too many resonances). Dotted lines in panels (d, e, f) show damping times for F -mode calculated using ordinary normal-fluid hydrodynamics (see the text for more details). Part of the mode IV (at $T^\infty < 6 \times 10^7$ K) is shown by dots in the panel (f), as described in the text.

7.4 Damping times for superfluid NSs

As in the case of eigenfrequencies, we first consider the e-folding times $\tau_{b+s}^{-1} \equiv \tau_{\text{bulk}}^{-1} + \tau_{\text{shear}}^{-1}$ for radial ($l = 0$) pulsations for the simplified model 1 of nucleon superfluidity (see Fig. 1).

In Fig. 7(a, d) we present the functions $\sigma(T^\infty)$ and $\tau_{b+s}(T^\infty)$, obtained using the approximate method of Sec. 6.2. The frequencies and damping times are plotted for normal F -mode (thick solid line) as well as for the first four superfluid modes $1, \dots, 4$ (dashed lines)¹⁶. In the region shaded in gray the function $\tau_{b+s}(T^\infty)$ for the normal mode was not plotted (there are too many merging resonances in this region). The dotted curve in Fig. 7(d, e, f) labeled F_{nfh} (‘nfh’ is the abbreviation for ‘normal-fluid hydrodynamics’) shows the damping time calculated using the ordinary hydrodynamics of nonsuperfluid liquid but taking into account the effects of superfluidity on the bulk and shear viscosities. This curve is analogous to the thick solid curve in Fig. 4, obtained under the same conditions but for the model 2 of nucleon superfluidity. The vertical dotted line in Fig. 7(a, d) indicates a temperature at which frequencies of normal F -mode and the first superfluid mode coincide.

We present a detailed analysis of Fig. 7(d) in what follows, together with description of the approximate solutions for nonradial oscillation modes (Figs. 8 and 9).

For comparison, Fig. 7(b, e) demonstrates the results of the exact calculation of frequencies $\sigma(T^\infty)$ and damping times $\tau_{b+s}(T^\infty)$ for the first four (I, \dots , IV) oscillation modes of the superfluid NS [the modes are shown by solid (I), dashed (II), dot-dashed (III), and dotted (IV) lines].

To see how well the approximate solution [Fig. 7(a, d)] agrees with the exact one [Fig. 7(b, e)], both solutions are presented in Fig. 7(c, f). Dashes correspond to approximate solution, solid lines – to exact solution. A portion of the mode IV in Fig. 7(f) is shown by thick dots because the corresponding approximate solution (the mode $1H$) is not plotted. One sees that the agreement between the approximate and exact solutions is reasonable everywhere (average error does not exceed 10 – 25%) except for the resonances (see below) and an interval of temperatures $T^\infty \lesssim 3 \times 10^7$ K where the mode III of exact solution

¹⁶ Notice that, in Figs. 7(a, b, c) we present, in logarithmic scale, parts of the spectra, which were already plotted in linear scale in Figs. 5(a, b, c), respectively.

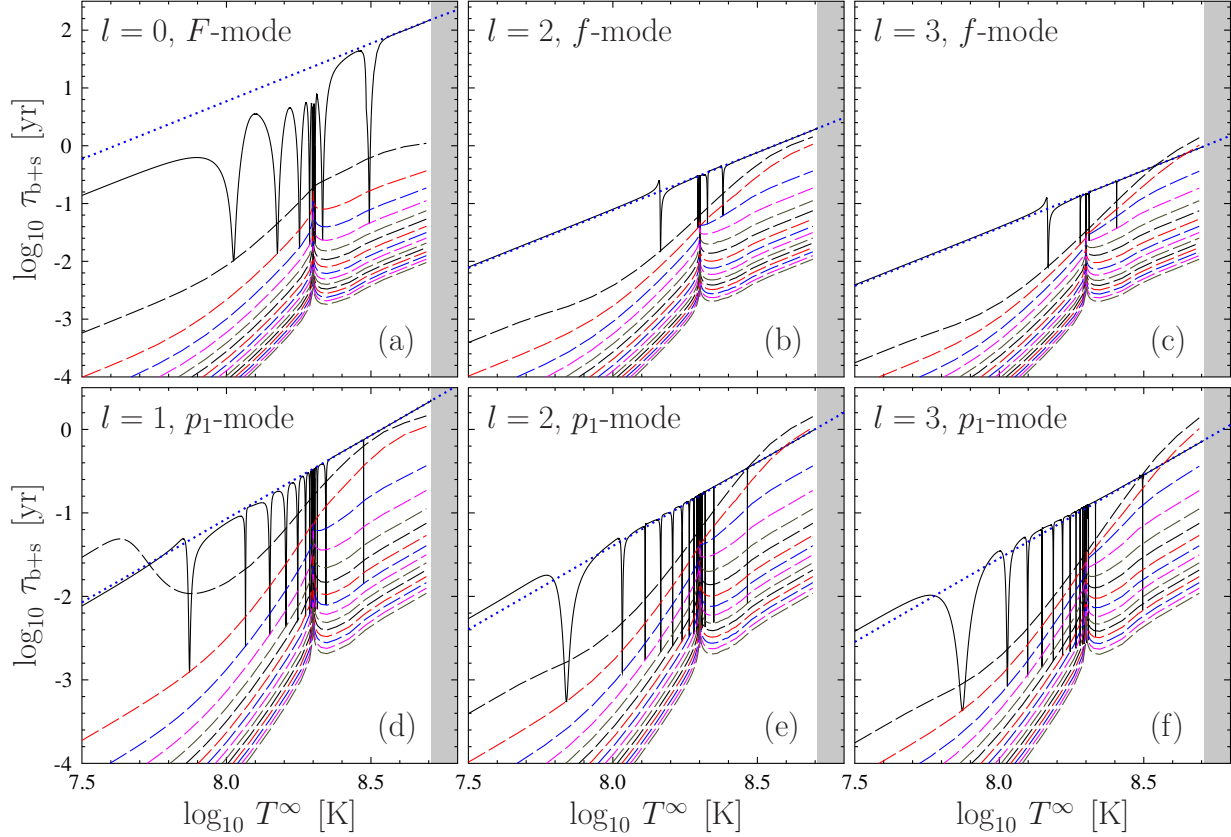


Figure 8. (color online) Damping times τ_{b+s} versus T^∞ for various oscillation modes for model 2 of nucleon superfluidity. On each panel we plot one normal mode (shown by solid line; its multipolarity and name is indicated) and first 15 superfluid modes (dashed lines). Dotted lines show $\tau_{b+s}(T^\infty)$ for normal modes calculated using normal-fluid hydrodynamics [see the text for more details]. In the shaded area all neutrons are normal and superfluid modes do not exist.

deviates from the second superfluid mode of approximate solution. To explain this deviation let us note that, as follows from Fig. 5(a), at such T^∞ the frequency of the normal mode $1H$ practically coincides with that of the second superfluid mode. In that case Eqs. (87) and (98) interact resonantly, so that the approximation of independent superfluid and normal modes is poor even though parameter s is small¹⁷.

Let us now consider the nonradial oscillations. Fig. 8 presents an approximate solution for the function $\tau_{b+s}(T^\infty)$, which is obtained for a realistic nucleon superfluidity model 2. By dashes we show superfluid modes, solid lines correspond to normal modes. Each panel in the figure is plotted for one normal mode (its name and multipolarity l are indicated) and for the first 15 superfluid modes with the same l . By dots, as in Fig. 7(d, e, f), we plot τ_{b+s} for a corresponding normal modes calculated using the ordinary normal-fluid hydrodynamics. In the shaded region superfluid modes were not plotted because all neutrons are normal there and the star oscillates as a nonsuperfluid.

In more detail damping times are demonstrated for quadrupole ($l = 2$) oscillation modes in Fig. 9. In particular, the normal p_1 -mode is shown there by solid lines. In the three lower panels we plot the dependence $\tau_{b+s}(T^\infty)$ in an increasingly larger scale. In the three upper panels we plot, in the same scale, the oscillation frequencies $\sigma(T^\infty)$ (the corresponding spectrum was already presented in Fig. 6 in linear scale). Left lower panel of Fig. 9 coincides with Fig. 8(e).

Let us discuss the main conclusions that can be drawn from the analysis of Figs. 7(d), 8, and 9.

1. For any normal mode the dependence $\tau_{b+s}(T^\infty)$ (solid lines in these figures), has a set of *resonance features* (spikes) concentrated (for radial and p -modes) to the critical temperature $T_{\text{cn}}^\infty(0)$ at which neutron superfluidity in the core centre dies out. For model 1 $T_{\text{cn}}^\infty(0) = T_{\text{cn max}}^\infty = 6 \times 10^8$ K (see Fig. 1), for model 2 $T_{\text{cn}}^\infty(0) \approx 2 \times 10^8$ K (see Fig. 2). The resonances appear when frequency of the normal mode approaches the frequency of one of the superfluid modes. For instance, solid line in Fig. 7(a) crosses superfluid modes four times [in Fig. 7(a, d) the temperature T^∞ of the first crossing is shown by the

¹⁷ For a quantitatively correct description of the function $\tau_{b+s}(T^\infty)$ in that case it is, in principle, straightforward to develop a perturbation theory similar to the degenerate perturbation theory of quantum mechanics (see below the discussion of resonances in Figs. 7, 8, and 9).

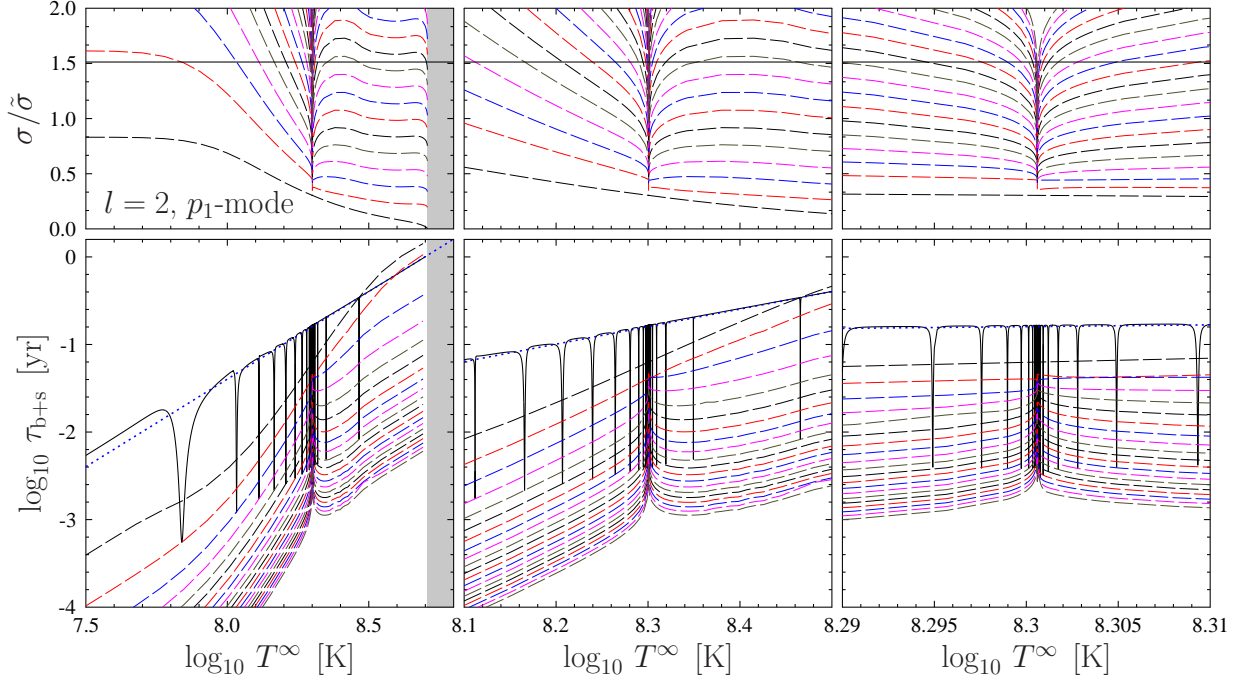


Figure 9. (color online) Eigenfrequencies σ (upper panels) and damping times τ_{b+s} (lower panels) versus T^∞ for quadrupole ($l = 2$) oscillation modes in an increasingly larger scale. The normal p_1 -mode is shown by solid lines. Left lower panel coincides with Fig. 8(e). Other lower panels are zoomed in versions of Fig. 8(e). Notations are the same as in Figs. 6 and 8.

vertical dotted line and equals $T^\infty \approx 10^8$ K]. Correspondingly, four resonances appear in Fig. 7(d). A similar situation can be observed in Figs. 8 and 9. Near resonances τ_{b+s} for normal mode rapidly decreases by 1–2 orders of magnitude (see item 2 below) and, in the resonance point, it becomes strictly equal to τ_{b+s} for the corresponding superfluid mode.

Such behavior of the approximate solution $\tau_{b+s}(T^\infty)$ for normal modes in the vicinity of resonances can be easily understood. In resonance points, in which the frequencies of superfluid and normal modes coincide, Eq. (98) has a nontrivial solution even in the absence of the source $\delta\mu_{\text{norm}l}$. For it to be satisfied with the source, the oscillation amplitude $\delta\mu$ must be infinitely large. In other words, in resonance points all the energy must be contained in superfluid degrees of freedom (in particular, near resonances $W_{\text{sfl}} \gg W_b$ and $V_{\text{sfl}} \gg V_b$). Formally, this means that in the resonance point the damping time τ_{b+s} should be exactly the same as for the superfluid mode.

Another important point that is worth noting is that, as follows from Fig. 7(f), the approximate solution for the normal radial F -mode describes *qualitatively well* the exact solution *near resonances* (the latter is shown by solid lines). We expect that the same is also true for nonradial modes for which the exact solution was not attempted. At first glance such an agreement between the approximate and exact solutions seems surprising because the approximation $s = 0$ should not work in the vicinity of resonances, where the frequencies of superfluid and normal modes are close to each other. Nevertheless, one verifies that this approximation is still suitable for a qualitatively correct description of the function $\tau_{b+s}(T^\infty)$ if one bears in mind that: (i) close to any resonance the exact solution is a linear superposition of independent solutions describing (intersecting) superfluid and normal modes and (ii) τ_{b+s} for the superfluid mode is much less than for the normal mode.

Items (i) and (ii) mean that, in the exact solution, the main contribution to τ_{b+s} comes from the superfluid mode (while the contribution from the normal mode is small). This leads us to conclusion that the superfluid modes are the main sources of viscous dissipation in the vicinity of resonance points. The same conclusion was already drawn above using the approximate method of Sec. 6.2. This explains why the approximate method gives qualitatively correct results for $\tau_{b+s}(T^\infty)$ near resonances.

In order to avoid confusion let us emphasize that the function $\tau_{b+s}(T^\infty)$ contains resonance features (spikes) for normal modes only in the approximate solution [see Figs. 7(d), 8, and 9]. In the exact solution any normal oscillation mode turns into a superfluid one near resonance (and vice versa). This leads to an abrupt decreasing (increasing) of τ_{b+s} and formation of a ‘step-like’ structure rather than spike [see Fig. 7(e)].

2. It was already mentioned above that, as follows from Figs. 7(d), 8, and 9, normal modes (far from resonances) damp out by 1–2 orders of magnitude slower than those superfluid modes with which they can have equal frequencies (i.e. intersect in the $\sigma - T^\infty$ plane).

What is the reason for such a fast damping of superfluid modes? To be more concrete, below we consider a low-temperature

case, $T^\infty \lesssim 3 \times 10^7$ K. There are three main factors: (i) For superfluid modes eigenfunctions W_{sfl} and V_{sfl} have a maximum in the central regions of a star where the shear viscosity is maximal. On the contrary, for normal modes the maximum of eigenfunctions W_{b} and V_{b} lies closer to the NS surface, where the shear viscosity coefficient can be substantially (5 and more times) smaller. As a consequence, $\mathfrak{W}_{\text{shear}}$ for superfluid modes turns out to be greater (and hence τ_{shear} smaller) than for normal modes. (ii) The energy of superfluid modes is given by Eq. (73) and depends on the quantity y [see Eq. (68) for the definition of y]. At low T^∞ the parameter y is small, $y \sim n_{\text{p}}/n_{\text{n}} \sim 0.04 \div 0.09$, which also results in decreasing of the characteristic damping times for superfluid modes¹⁸. (iii) This factor is particularly important for radial oscillations ($l = 0$) and is related to a coefficient α_1 in the expression (78) for the damping time τ_{shear} due to shear viscosity. This coefficient is given by Eq. (81) which is a sum of four terms. It turns out that for the normal radial modes the first term is well compensated by the third term H_2 , while the other terms vanish. For the superfluid modes such compensation does not occur because for them $H_2 = 0$.

3. At low enough T^∞ the damping times for normal radial and p -modes can be several times larger or smaller than $\tau_{\text{b+s}}$, calculated employing ordinary hydrodynamics of nonsuperfluid liquid but accounting for the effects of superfluidity on the bulk and shear viscosity coefficients (dotted lines in Figs. 7(d), 8(a, d, e, f), and 9). Let us inspect, for example, Fig. 8(a). One sees that at $T^\infty \lesssim 10^8$ K $\tau_{\text{b+s}}$, calculated in the frame of nonsuperfluid hydrodynamics, is approximately 4 times larger than $\tau_{\text{b+s}}$ determined self-consistently. This difference arises because to plot the dotted curve we used the formulas of Sec. 4 in which $W_{\text{sfl}} = V_{\text{sfl}} = 0$. As T^∞ grows, however, the difference in two ways of calculating $\tau_{\text{b+s}}$ rapidly decreases because the SFL-region becomes smaller and hence its contribution to $\tau_{\text{b+s}}$ becomes less and less pronounced.

4. Unlike the radial and p -modes, the agreement between dotted and solid lines for normal f -modes is very good [see Fig. 8(b, c)], which means that for these modes use of the nonsuperfluid hydrodynamics (far from resonances) is well justified. The reason for such a good agreement of damping times is related to a relatively weak compression-decompression of matter in the course of the f -type oscillations. As a consequence, for the normal f -modes the source $\delta\mu_{\text{norm } l}$ in Eq. (98) is small, so that far from the resonances $\delta\mu \approx 0$ and the superfluid degrees of freedom are almost not excited [$W_{\text{sfl}} \approx V_{\text{sfl}} \approx 0$, see Eqs. (48), (52), and (90)]. This result confirms, extends and, we think, provides a deeper understanding, of the results previously obtained in a Newtonian framework by, e.g., Lindblom & Mendell (1994) and Andersson et al. (2009).

5. At $T^\infty \rightarrow T_{\text{cn max}}^\infty = 6 \times 10^8$ K one can observe the rapid increasing of $\tau_{\text{b+s}}$ for superfluid modes in Fig. 7. It is bounded from above by τ_{bulk} and is related with the tendency of τ_{shear} to grow to infinity in this limit. Such a behaviour of τ_{shear} was discussed in detail in Kantor & Gusakov (2011) and is specific for model 1 of nucleon superfluidity.

8 SUMMARY

In this paper we, for the first time, self-consistently analyze the effects of nucleon superfluidity on damping of oscillations of nonrotating *general relativistic* NSs. Our main results are summarized below.

1. The analytic formulas are derived for the oscillation energy E_{mech} (71) and for the characteristic damping times τ_{bulk} (77) and τ_{shear} (78) due to the bulk and shear viscosities. These expressions are valid for oscillations of arbitrary multipolarity l . The expression (71) for E_{mech} is the generalization of the formula (26) of Thorne & Campolattaro (1967), written for a nonsuperfluid NS. The expressions (77) and (78) are the generalizations, to the case of superfluidity, of the formulas (5) and (6) in Cutler et al. (1990). Notice that the damping times, calculated using the formulas of Cutler et al. (1990) appear to be 2 times smaller than our τ_{bulk} and τ_{shear} , calculated from Eqs. (77) and (78) under assumption that superfluid degrees of freedom are suppressed (i.e., $W_{\text{sfl}} = V_{\text{sfl}} = 0$).

2. An approximate method is developed in detail and applied, which allows one to easily determine the eigenfrequencies and eigenfunctions of an oscillating superfluid NS, provided that they are known for a normal (nonsuperfluid) star of the same mass (see Sec. 6.2). The method is based on the approximate decoupling of equations describing superfluid and normal oscillation modes and exploits the ideas first formulated in Gusakov & Kantor (2011); Chugunov & Gusakov (2011).

3. Using radial oscillations as an example, and adopting the simplified model 1 of nucleon superfluidity (Fig. 1), we demonstrate that this method leads to oscillation frequencies and characteristic damping times that agree well with the results of exact calculation.

4. The approximate method of Sec. 6.2 is applied to study nonradial oscillations of a superfluid NS assuming the realistic model 2 of nucleon superfluidity (Fig. 2). A number of normal and superfluid oscillation modes with multipolarities $l = 0, \dots, 3$ are considered. In particular, the following normal modes are analyzed: F -mode for $l = 0$, p_1 -mode for $l = 1$, f - and p_1 -modes for $l = 2$ and 3.

It is demonstrated that:

(i) As a rule, for any given normal mode (whose frequency σ coincides with the corresponding frequency of a nonsuperfluid

¹⁸ To get an estimate for y we made use of the sum rule $\mu_{\text{n}} Y_{\text{nn}} + \mu_{\text{p}} Y_{\text{np}} = n_{\text{n}}$ valid at $T^\infty = 0$ (Gusakov, Kantor & Haensel 2009a), and neglected the small matrix element Y_{np} in comparison with Y_{nn} .

NS and does not depend on the internal redshifted stellar temperature T^∞) the viscous damping time $\tau_{b+s} \equiv (\tau_{\text{bulk}}^{-1} + \tau_{\text{shear}}^{-1})^{-1}$ is one order of magnitude *greater* than τ_{b+s} for those superfluid modes that can intersect the normal mode in the $\sigma - T^\infty$ plane. This effect is non-local (occurs only after integration over the NS volume) and is determined by a number of factors (see item 2 of Sec. 7.4).

(ii) The function $\tau_{b+s}(T^\infty)$ for any normal mode contains *resonance features*. In resonance points the frequency σ of a normal mode coincides with that of some of the superfluid modes (their σ depend on T^∞). When passing a resonance (e.g., with growing T^∞), τ_{b+s} initially rapidly decreases (by 1–2 orders of magnitude) until it reaches the value of τ_{b+s} for this superfluid mode and, after that, it increases again (see Figs. 7(d), 8, and 9).

(iii) Resonance features (spikes) appear only in the approximate treatment of Sec. 6, in which the normal and superfluid modes intersect at resonance points [see, e.g., Fig. 5(a)]. In the exact solution instead of crossings one has *avoided crossings* of modes [Fig. 5(b)]. Near avoided crossings any real mode changes its behaviour from normal-like to superfluid-like (and vice versa). As a result, instead of spikes one has a very rapid step-like decreasing (increasing) of τ_{b+s} [cf. Figs. 7(d) and 7(e)].

(iv) Sufficiently far from the resonances τ_{b+s} for normal radial and p -modes, determined self-consistently employing the hydrodynamics of a superfluid liquid, can differ several fold from τ_{b+s} , calculated using the ordinary normal-fluid hydrodynamics (but accounting for the effects of superfluidity on the shear and bulk viscosities). The latter approximation is often adopted in the literature devoted to oscillations of NSs.

(v) In contrast to radial and p -modes, for f -modes far from the resonances, use of the ordinary hydrodynamics of nonsuperfluid liquid for calculation of τ_{b+s} is well justified. The reason is that for f -type oscillations the imbalance $\delta\mu$ of chemical potentials is relatively small (matter does not compress significantly during oscillations). Thus, superfluid degrees of freedom are almost not excited (see Secs. 6.2 and 7.4).

(vi) Since for f -modes far from the resonances $\delta\mu$ is small (that is, deviation from the beta-equilibrium is weak), bulk viscous damping of f -modes is suppressed in comparison to p -modes.

Though here we only considered oscillations of superfluid nonrotating NSs, we expect that the main conclusions of this work will also remain (mostly) unchanged for rotating NSs. Our results indicate that dissipative evolution of oscillating NSs may follow quite different scenarios than those usually considered in the literature. This is especially true if one is interested in the combined analysis of damping of oscillations and thermal evolution of a NS or in the analysis of instability windows, that is the values of T^∞ and rotation frequency at which a star becomes unstable with respect to the emission of gravitational waves (e.g., the r -mode instability, see Andersson 1998; Friedman & Morsink 1998). These issues are extremely interesting and important, but we left them beyond the scope of the present paper and will address the related topics in our subsequent publication.

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APPENDIX A: BOUNDARY CONDITIONS TO EQUATION (98)

Equation (98) should be solved in the region of a NS core where neutrons are superfluid (SFL-region). If the NS centre is occupied by the neutron superfluidity, then for regularity of the solution at $r \rightarrow 0$ it is necessary that

$$\delta\mu_l \propto r^l. \quad (\text{A1})$$

The conditions at the boundary of the SFL-region follow from the requirement of the absence of particle transfer (baryons and electrons) through the interface. One obtains from the definitions (4)–(7)

$$\mathbf{X}_\perp = 0, \quad (\text{A2})$$

where \mathbf{X}_\perp is the component of the vector X^j perpendicular to the interface. To rewrite Eq. (A2) in terms of $\delta\mu_l(r)$, it is necessary to consider two possibilities:

(i) The boundary (one of the boundaries) between the SFL-region and nonsuperfluid matter lies inside the core and is defined by the condition $T = T_{\text{cn}}(R_b)$ [R_b is the radial coordinate of the boundary]. Then at the boundary $Y_{\text{nn}}(R_b) = Y_{\text{np}}(R_b) = 0$ and from Eqs. (90) and (98) one has

$$\delta\mu'_l = \frac{e^{\lambda-\nu/2} \omega^2}{h' \mathfrak{B}} (\delta\mu_l - \delta\mu_{\text{norm}l}). \quad (\text{A3})$$

(ii) Outer boundary of the SFL-region coincides with the crust-core interface ($R_b = R_{cc}$). In that case $T < T_{cn}(R_{cc})$ [that is $Y_{nn}(R_{cc})$ and $Y_{np}(R_{cc})$ are non-zero] and from Eq. (90) it follows that

$$\delta\mu'_l(R_{cc}) = 0. \quad (\text{A4})$$

The conditions (A1)–(A4) are necessary and sufficient for solving Eq. (98).

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